

# MULTI-BUMP SOLUTIONS FOR FRACTIONAL NIRENBERG PROBLEM

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ABSTRACT. We consider the multi-bump solutions of the following fractional Nirenberg problem

$$(-\Delta)^s u = K(x)u^{\frac{n+2s}{n-2s}}, \quad u > 0 \quad \text{in } \mathbb{R}^n, \quad (0.1)$$

where  $s \in (0, 1)$  and  $n > 2+2s$ . If  $K$  is a periodic function in some  $k$  variables with  $1 \leq k < \frac{n-2s}{2}$ , we proved that (0.1) has multi-bump solutions with bumps clustered on some lattice points in  $\mathbb{R}^k$  via Lyapunov-Schmidt reduction. It is also established that the equation (0.1) has an infinite-many-bump solutions with bumps clustered on some lattice points in  $\mathbb{R}^n$  which is isomorphic to  $\mathbb{Z}_+^k$ .

## 1. INTRODUCTION AND MAIN RESULTS

The classic Nirenberg problem asks that on the standard sphere  $(\mathbb{S}^n, g_{\mathbb{S}^n})$  with  $n \geq 2$ , whether there exists a function  $w$  such that the scalar curvature (Gauss curvature in the dimension 2) of the conformal metric  $g = e^w g_{\mathbb{S}^n}$  equals to a prescribed function  $\tilde{K}$ . This problem is equivalent to solving the following equations

$$-\Delta_{g_{\mathbb{S}^n}} w + 1 = \tilde{K} e^{2w} \quad \text{on } \mathbb{S}^2 \quad (1.1)$$

and

$$-\Delta_{g_{\mathbb{S}^n}} v + \frac{n-2}{4(n-1)} R_{g_{\mathbb{S}^n}} v = \frac{n-2}{4(n-1)} \tilde{K} v^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n \quad \text{for } n \geq 3, \quad (1.2)$$

where  $R_{g_{\mathbb{S}^n}} = n(n-1)$  is the scalar curvature of  $(\mathbb{S}^n, g_{\mathbb{S}^n})$  and  $v = e^{\frac{n-2}{4}w}$ . The linear operators defined on left-hand side of the equation (1.1) and (1.2) are called the conformal Laplacian on  $\mathbb{S}^n$ .

For any Riemannian manifold  $(M, g)$ , the conformal Laplacian is defined by  $P_1^g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$ , where  $R_g$  is the scalar curvature of  $(M, g)$ . Let  $u > 0$  and  $h = u^{\frac{4}{n-2}} g$ , the conformal Laplacian has the following conformally invariant property

$$P_1^g(u\phi) = u^{\frac{n+2}{n-2}} P_1^h(\phi) \quad \text{for } \phi \in C^\infty(M).$$

The Paneitz operator  $P_2^g$  is another interesting conformal invariant operator. It was defined in [27] by

$$P_2^g = \Delta_g^2 + \text{div}_g(a_n R_g \text{Id} - b_n \text{Ric}_g) \nabla_g + \frac{n-4}{2} Q_g,$$

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where  $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$ ,  $b_n = -\frac{4}{n-2}$ ,  $\mathcal{R}ic : TM \rightarrow TM$  is a (1,1)-tensor operator defined by  $\mathcal{R}ic_i^j = g^{jk}\mathcal{R}ic_{ki}$  and  $Q_g = -\frac{2}{(n-2)^2}|\mathcal{R}ic_g|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2 - \frac{1}{2(n-1)}\Delta_g R_g$  which is called the  $Q$ -curvature of  $(M, g)$ .

Later on, more conformally covariant elliptic operators were found. The operator  $P_1^g$  and  $P_2^g$  were generalized by Graham, Jenne, Mason and Sparling in [16] to a sequence of integer order conformally covariant elliptic operators  $P_k^g$  for  $k \in \mathbb{N}_+$  if  $n$  is odd; and  $k \in \{1, \dots, \frac{n}{2}\}$  if  $n$  is even. Furthermore, any real number order conformally covariant pseudo-differential operator was intrinsically defined by Peterson in [28]. Graham and Zworski in [17] proved that the operators  $P_k^g$  can be considered as the residue of a meromorphic family of scattering operators  $S(s)$  at  $s = \frac{n}{2} + k$ . Then a family of non-integer order conformally covariant pseudo-differential operators  $P_s^g$  ( $0 < s < \frac{n}{2}$ ) were naturally defined. Using the localization method in [5], Chang and González [6] showed that for any  $s \in (0, \frac{n}{2})$ , the operator  $P_s^g$  can also be defined as a Dirichlet-to-Neumann operator of a conformally compact Einstein manifold.

The conformally covariant law for  $P_s^g$  means that for any Riemannian manifold  $(M, g)$  and a conformal transformation  $h = v^{\frac{4}{n-2s}}g$ ,  $v > 0$ , there holds

$$P_s^g(v\phi) = v^{\frac{n+2s}{n-2s}}P_s^h(\phi) \quad \text{for } \phi \in C^\infty(M).$$

Especially,  $P_s^g(1)$  is called the  $Q_s$  curvature or  $s$ -curvature of  $(M, g)$  (see [6] and [18] for example).

The fractional Nirenberg problem was naturally raised on  $Q_s$  curvature, it asks that on the standard sphere  $\mathbb{S}^n$ , whether there exists a function  $v > 0$  such that the  $Q_s$  curvature of the conformal metric  $g = v^{\frac{4}{n-2s}}g_{\mathbb{S}^n}$  equals to a prescribed function  $\tilde{K}$ . It can be reduced to the existence of the solution of the following equation

$$P_s^{g_{\mathbb{S}^n}}(v) = \tilde{K}v^{\frac{n+2s}{n-2s}}, \quad v > 0 \quad \text{on } \mathbb{S}^n, \quad (1.3)$$

where  $s \in (0, 1)$ ,  $n > 2s$  and  $\tilde{K}$  is a given positive function.

It was shown in [4] that the operator  $P_s^{g_{\mathbb{S}^n}}$  is an intertwining operator and can be expressed as

$$P_s^{g_{\mathbb{S}^n}} = \frac{\Gamma(B + \frac{1}{2} + s)}{\Gamma(B + \frac{1}{2} - s)}, \quad B = \sqrt{-\Delta_{\mathbb{S}^n} + \left(\frac{n-1}{2}\right)^2},$$

where  $\Delta_{\mathbb{S}^n}$  is the Beltrami-Laplacian operator. What is more,  $P_s^{g_{\mathbb{S}^n}}$  is more concrete under the stereographic projection. Let

$$F : \mathbb{R}^n \rightarrow \mathbb{S}^n \setminus \{\mathcal{N}\}, \quad x \mapsto \left( \frac{2x}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1} \right)$$

be the inverse of stereographic projection, where  $\mathcal{N}$  is the north pole of  $\mathbb{S}^n$ . Then it holds that

$$P_s^{g_{\mathbb{S}^n}}(\phi) \circ F = |J_F|^{-\frac{n+2s}{2n}}(-\Delta)^s(|J_F|^{\frac{n-2s}{2n}}(\phi \circ F)),$$

where  $(-\Delta)^s$  is the fractional Laplacian defined by

$$(-\Delta)^s \phi(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x - y|^{n+2s}} dy, \quad \text{where } \phi \in C^\infty(\mathbb{R}^n).$$

If we write  $u = |J_F|^{\frac{n-2s}{2n}}(v \circ F)$  and  $K = \tilde{K} \circ F$ , the equation (1.3) is transformed into

$$(-\Delta)^s u = K(x)u^{\frac{n+2s}{n-2s}}, \quad u > 0 \quad \text{in } \mathbb{R}^n, \quad (1.4)$$

where  $s \in (0, 1)$  and  $n > 2s$ . The existence of the solutions to the problem (1.3) has been proved under various conditions (see for example [1, 2, 7, 9–11, 18, 19]). The compactness of the solutions to (1.3) was studied in [18]. Chen and Zheng [10] found a 2-peak solution when  $K(x) = 1 + \varepsilon \tilde{K}(x)$  has at least two critical points and satisfies some local conditions. What is more, Liu in [25] constructed infinitely many 2-peak solutions when  $K$  has a sequence of strictly local maximum points moving to infinity. When  $K$  is a radial symmetric function, in [24] and [26] it was showed that (1.4) has infinitely many non-radial solutions.

In this paper, we continue to study the bump solutions or peak solutions of (1.4). Assume that  $K$  satisfies the following conditions

- ( $H_1$ )  $0 < \inf_{\mathbb{R}^n} K \leq \sup_{\mathbb{R}^n} K < \infty$ ;
- ( $H_2$ )  $K(x)$  is a  $C^{1,1}$  function, and 1-periodic in the first  $k$  variables  $x_1, \dots, x_k$ ;
- ( $H_3$ ) 0 is a critical point of  $K$ , and in a neighborhood of 0, there is a number  $\beta \in (n - 2s, n)$  such that

$$K(x) = K(0) + \sum_{i=1}^n a_i |x_i|^\beta + R(x),$$

where  $a_i \neq 0$  for  $i = 1, \dots, n$ ,  $\sum_{i=1}^n a_i < 0$ ,  $R(y) \in C^{[\beta]-1,1}$  and  $\sum_{j=0}^{[\beta]} |\nabla^j R(y)| |y|^{-\beta+j} = o(1)$  as  $y \rightarrow 0$ . Here  $\nabla^j R(y)$  denote all of the possible derivatives of  $R(y)$  of the order  $j$ .

We note that the conditions ( $H_1$ )-( $H_3$ ) and the condition ( $\mathcal{K}$ ) in [25] has some intersecion. When  $K$  satisfies both the condition ( $\mathcal{K}$ ) in [25] and our conditions ( $H_1$ )-( $H_3$ ), the equation (1.4) has infinitely many 2-peak solutions according to [25].

In this paper, we will show that equation (1.4) has solutions with large number bumps and its bumps located near some lattice points in  $\mathbb{R}^k$  with  $1 \leq k < \frac{n-2s}{2}$ .

Let  $Q_m := ([0, m+1)^k \times \mathbf{0}) \cap \mathbb{Z}^n$ , where  $m \in \mathbb{N}_+ \cup \{\infty\}$  and  $\mathbf{0}$  is a zero vector in  $\mathbb{R}^{n-k}$ .

**Theorem 1.1.** *Suppose  $n > 2s + 2$ ,  $m \in \mathbb{N}_+ \cup \{\infty\}$  and  $1 \leq k < \frac{n-2s}{2}$ . If  $K$  satisfies the conditions ( $H_1$ )-( $H_3$ ), there exists an integer  $l_0 \in \mathbb{N}$ , such that for any integer  $l > l_0$ , the equation (1.4) has a solution with its bumps clustered on  $lQ_m$ .*

Notice that  $Q_\infty$  is an infinite lattice which isomorphic to  $\mathbb{Z}_+^k$ . So we get an infinite-many-bump solution of the equation (1.5) via Theorem 1.1.

In order to prove Theorem 1.1, we assume  $K(0) = 1$  with no loss of generality. For any positive integer  $l$ , define  $\lambda = l^{\frac{n-2s}{\beta-n+2s}}$ . Then we have  $\lambda^\beta = (\lambda l)^{n-2s}$ . Using the transformation  $u(x) \mapsto \lambda^{-\frac{n-2s}{2}} u(\frac{x}{\lambda})$ , we can change the equation (1.4) into

$$(-\Delta)^s u = K\left(\frac{x}{\lambda}\right) u^{\frac{n+2s}{n-2s}}, \quad u > 0, \quad \text{in } \mathbb{R}^n. \quad (1.5)$$

The functional corresponding to equation (1.5) is

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{n-2s}{2n} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) (u_+)^{\frac{2n}{n-2s}}, \quad u \in \dot{H}^s(\mathbb{R}^n),$$

where  $u_+ = \max\{u, 0\}$ . The Hilbert space  $\dot{H}^s(\mathbb{R}^n)$  is the completion of  $C_0^\infty(\mathbb{R}^n)$  under the Gagliardo semi-norm (cf. [15] for detail)

$$[u]_{\dot{H}^s(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \right)^{\frac{1}{2}} = \left( 2C(n, s) \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 \right)^{\frac{1}{2}},$$

where  $C(n, s)$  is a constant depending on  $n$  and  $s$ . It is well known that  $\dot{H}^s(\mathbb{R}^n)$  can be imbedded into  $L^{2^*(s)}(\mathbb{R}^n)$  and the following Hardy-Littlewood-Sobolev inequality holds

$$S \left( \int_{\mathbb{R}^n} |u|^{2^*(s)} \right)^{\frac{2}{2^*(s)}} \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2, \quad \text{for } u \in C_0^\infty(\mathbb{R}^n), \quad (1.6)$$

where  $s \in (0, 1)$ ,  $n > 2s$  and  $2^*(s) = \frac{2n}{n-2s}$ . Lieb [23] proved that the extremals corresponding to the best constant  $S$  of (1.6) are of the form

$$U_{\xi, \Lambda, C_0} = C_0 \left( \frac{\Lambda}{1 + \Lambda^2 |x - \xi|^2} \right)^{\frac{n-2s}{2}},$$

where  $C_0, \Lambda \in \mathbb{R}_+$  and  $\xi \in \mathbb{R}^n$ .

Choosing a suitable constant  $C_0 = C_0(n, s)$ , we see that the function  $U_{\xi, \Lambda} := U_{\xi, \Lambda, C_0}$  solves the equation

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}}, \quad u > 0 \text{ in } \mathbb{R}^n. \quad (1.7)$$

Under some decay assumptions, [8, 20, 22] proved that all the solutions of (1.7) are only of the form  $U_{\xi, \Lambda}$ . Furthermore, it was proved in [13] that the solution  $U_{\xi, \Lambda}$  of the equation (1.7) is nondegenerate, *i.e.* any bounded solution of the equation  $(-\Delta)^s \phi = \frac{n+2s}{n-2s} U_{\xi, \Lambda}^{\frac{4s}{n-2s}} \phi$  is a linear combination of  $\frac{\partial U_{\xi, \Lambda}}{\partial \Lambda}$  and  $\frac{\partial U_{\xi, \Lambda}}{\partial \xi_i}$ ,  $i = 1, 2, \dots, n$ .

We will use the functions  $U_{\xi, \Lambda}$  to construct the approximate solutions of the equation (1.5). We define  $X_{l,m} = \{\lambda x | x \in Q_m\}$  and arrange it in any way as a sequence  $X_{l,m} = \{X^i\}_{i=1}^{(m+1)^k}$ . Let  $P^i \in B_{\frac{1}{2}}(X^i) = \{X \in \mathbb{R}^n | |X - X^i| < \frac{1}{2}\}$ ,  $\Lambda_i \in [C_1, C_2]$ , for  $i = 1, 2, \dots, (m+1)^k$ , where  $C_1$  and  $C_2$  are some positive numbers to be defined later (see (3.6)). Let

$$W_m(x) := \sum_{i=1}^{(m+1)^k} U_{P^i, \Lambda_i}(x)$$

to be an approximate solution of the problem (1.5).

**Theorem 1.2.** *Under the same conditions of Theorem 1.1, there exists an interger  $l_0 > 0$ , such that for any integer  $l > l_0$ , equation (1.5) has a  $C_{loc}^2$  solution  $u_m$  of the form*

$$u_m = W_m + \phi_m,$$

where  $m \in \mathbb{N}_+ \cup \{\infty\}$ ,  $|\phi|_{L^\infty(\mathbb{R}^n)} \rightarrow 0$  and  $\max_{i=1, \dots, (m+1)^k} \{|P^i - X^i|\} \rightarrow 0$  as  $l \rightarrow \infty$ .

As a consequence of Theorem 1.2, we have

**Corollary 1.3.** *Under the same conditions of Theorem 1.1, the equation (1.4) has infinitely many multi-bump solutions.*

Theorem 1.1 follows from Theorem 1.2. So we only need to prove Theorem 1.2. In this article we use  $l$  as the perturbation parameter and follow the methods developed in [21, 29]. In the section 2, we carry out the Liapunov-Schmidt reduction. Theorem 1.2 is proved in the section 3. Some useful estimations are presented in Appendix A. The expansions of the functional  $\frac{\partial}{\partial \Lambda_i} I(W_m)$  and  $\frac{\partial}{\partial P_j^i} I(W_m)$  are shown in Appendix B.

In this article,  $C$  denotes a varying constant independent of  $m$ .

## 2. FINITE DIMENSIONAL REDUCTION

In this section, we will carry out the Lyapunov-Schmidt reduction in the case of  $m < \infty$ . We define two weighted norms

$$\|u\|_* = \sup_{y \in \mathbb{R}^n} \left( \gamma(y) \sum_{i=1}^{(m+1)^k} \frac{1}{(1 + |y - X^i|)^{\frac{n-2s}{2} + \tau}} \right)^{-1} |u(y)|,$$

and

$$\|u\|_{**} = \sup_{y \in \mathbb{R}^n} \left( \gamma(y) \sum_{i=1}^{(m+1)^k} \frac{1}{(1 + |y - X^i|)^{\frac{n+2s}{2} + \tau}} \right)^{-1} |u(y)|,$$

where

$$\gamma(y) = \min \left\{ \min_{i=1, \dots, (m+1)^k} \left( \frac{1 + |y - X^i|}{\lambda} \right)^{\tau-s}, 1 \right\},$$

and  $\tau \in (k, \frac{n-2s}{2})$  is a constant.

Consider the following equation

$$(-\Delta)^s \phi - \frac{n+2s}{n-2s} K \left( \frac{x}{\lambda} \right) W_m^{\frac{4s}{n-2s}} \phi = g \quad \text{in } \mathbb{R}^n. \quad (2.1)$$

**Lemma 2.1.** *Let  $\phi$  be a solution of the equation (2.1), then we have the following estimate*

$$\begin{aligned} & \left( \gamma(y) \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}} \right)^{-1} |\phi(y)| \\ & \leq C \|g\|_{**} + C \|\phi\|_* \left( \frac{1}{(\lambda l)^{\frac{4s}{n-2s}k}} \frac{\sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau + \theta}}}{\sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}}} \right), \end{aligned} \quad (2.2)$$

where  $\theta > 0$  is a constant and  $C$  is independent of  $m$ .

*Proof.* We rewrite the equation (2.1) into an integral equation

$$\phi(y) = C_1(n, s) \int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} \left( \frac{n+2s}{n-2s} K \left( \frac{z}{\lambda} \right) W_m^{\frac{4s}{n-2s}}(z) \phi(z) + g(z) \right) dz, \quad (2.3)$$

where the constant  $C_1(n, s)$  is defined in the Green function of  $(-\Delta)^s$  on  $\mathbb{R}^n$  (cf. [5]).

From Lemma A.5, we get

$$\begin{aligned}
& \left| \int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} K\left(\frac{z}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(z) \phi(z) dz \right| \\
& \leq C \|\phi\|_* \int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} W_m^{\frac{4s}{n-2s}}(z) \gamma(z) \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|z-X^h|)^{\frac{n-2s}{2}+\tau}} dz \\
& \leq C \|\phi\|_* \left( \gamma(y) \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau+\theta}} \right. \\
& \quad \left. + \frac{1}{(\lambda l)^{\frac{4s}{n-2s}k}} \gamma(y) \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau}} \right). \tag{2.4}
\end{aligned}$$

For the second term on the right-hand side of (2.3), we have

$$\int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} |g(z)| dz \leq \|g\|_{**} \int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} \gamma(z) \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|z-X^h|)^{\frac{n+2s}{2}+\tau}} dz. \tag{2.5}$$

Using Lemma A.2, we obtain

$$\int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|z-X^h|)^{\frac{n+2s}{2}+\tau}} dz \leq C \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|z-X^h|)^{\frac{n-2s}{2}+\tau}}. \tag{2.6}$$

Define  $B_i := B_{\lambda l}(X^i)$ ,  $B_{i,m} := B_{r_0}(X^i)$  with  $r_0 = \max\{\frac{m}{4}, 1\}$  and

$$\Omega_i := \{z \in \mathbb{R}^n : |z-X^i| = \min_{j=1, \dots, (m+1)^k} |z-X^j|\}.$$

Without loss of generality, we assume  $y \in \Omega_1$ . Make use of Lemma A.3 under different cases, we have

$$\frac{1}{\lambda^{\tau-s}} \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|y-X^h|)^{\frac{n}{2}}} \leq C \left( \frac{1+|y-X^1|}{\lambda} \right)^{\tau-s} \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau}}. \tag{2.7}$$

Using Lemma A.2 and (2.7), we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} \min_{i=1, \dots, (m+1)^k} \left\{ \left( \frac{1+|z-X^i|}{\lambda} \right)^{\tau-s} \right\} \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|z-X^h|)^{\frac{n+2s}{2}+\tau}} dz \\
& \leq \frac{C}{\lambda^{\tau-s}} \int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|z-X^h|)^{\frac{n}{2}+2s}} dz \\
& \leq \frac{C}{\lambda^{\tau-s}} \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|y-X^h|)^{\frac{n}{2}}} \\
& \leq C \left( \frac{1+|y-X^1|}{\lambda} \right)^{\tau-s} \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau}}. \tag{2.8}
\end{aligned}$$

From the definition of  $\gamma(y)$ , (2.5), (2.6) and (2.8), we know

$$\int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2s}} |g(z)| dz \leq C \|g\|_{**} \gamma(y) \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau}}. \quad (2.9)$$

Now (2.2) follows from (2.3), (2.4) and (2.9).  $\square$

Consider the following problem

$$\begin{cases} (-\Delta)^s \phi - \frac{n+2s}{n-2s} K(\frac{x}{\lambda}) W_m^{\frac{4s}{n-2s}} \phi = g + \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j}, \\ \int_{\mathbb{R}^n} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} \phi dx = 0, \quad \phi \in \dot{H}^s(\mathbb{R}^n), \quad i = 1, \dots, (m+1)^k, \quad j = 1, \dots, n+1, \end{cases} \quad (2.10)$$

where  $Z_{i,j} = \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}$  for  $j = 1, \dots, n$  and  $Z_{i,n+1} = \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i}$ .

**Lemma 2.2.** *Assume  $\phi$  solves the problem (2.10), there is exists  $l_0 > 0$ , such that for all  $l > l_0$ , we have  $\|\phi\|_* \leq C \|g\|_{**}$ , where  $C$  is independent of  $m$ .*

*Proof.* If this lemma is not right, then there would be sequences  $\{g_l\}_{l=1}^\infty$  and  $\{\phi_l\}_{l=1}^\infty$  satisfying (2.10) with  $\|\phi_l\|_* = 1$  and  $\|g_l\|_{**} \rightarrow 0$  as  $l \rightarrow +\infty$ . For notation simplicity, we suppress  $l$  in the argument below.

First, we give an estimate of the parameters  $c_{ij}^{(m)}$ . Multiplying (2.10) with  $Z_{r,t}$  and integrating on both sides, we get

$$-\frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K(\frac{x}{\lambda}) W_m^{\frac{4s}{n-2s}} \phi Z_{r,t} dx = \int_{\mathbb{R}^n} g Z_{r,t} dx + \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} \int_{\mathbb{R}^n} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} Z_{r,t} dx. \quad (2.11)$$

For the first term on the right hand side of (2.11), using Lemma A.2, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} g Z_{r,t} dx \right| &\leq C \|g\|_{**} \int_{\mathbb{R}^n} \frac{1}{(1+|x-X^r|)^{n-2s}} \gamma(x) \sum_{j=1}^{(m+1)^k} \frac{1}{(1+|x-X^j|)^{\frac{n+2s}{2}+\tau}} dx \\ &\leq \frac{C}{\lambda^{\tau-s}} \|g\|_{**} \int_{\mathbb{R}^n} \frac{1}{(1+|x-X^r|)^{n-2s}} \sum_{j=1}^{(m+1)^k} \frac{1}{(1+|x-X^j|)^{\frac{n}{2}+2s}} dx \\ &\leq C \frac{\|g\|_{**}}{\lambda^{\tau-s}} \left( \int_{\mathbb{R}^n} \frac{1}{(1+|x-X^r|)^{n+\frac{n}{2}}} dx + \sum_{j \neq r} \frac{1}{|X^j - X^r|^{\frac{n}{2}}} \right) \\ &\leq C \frac{\|g\|_{**}}{\lambda^{\tau-s}}, \end{aligned}$$

where we have used the fact that

$$\sum_{j \neq r} \frac{1}{|X^j - X^r|^{\frac{n}{2}}} \text{ converges for } \frac{n}{2} > k. \quad (2.12)$$

Since the left hand side of the equation (2.11) is estimated in Lemma A.6, we have

$$\sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} \int_{\mathbb{R}^n} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} Z_{r,t} dx = \frac{1}{\lambda^{\tau-s}} O \left( \|g\|_{**} + \frac{\|\phi\|_*}{(\lambda l)^{\frac{n}{2}}} \right).$$

As we know  $\int_{\mathbb{R}^n} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} Z_{i,t} dx = C \delta_{jt}$  and  $\int_{\mathbb{R}^n} |U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} Z_{r,t}| dx \leq \frac{C}{|X^i - X^r|^{n-2s}}$  for  $i \neq r$ , we obtain

$$\max_{i,j} \{ |c_{ij}^{(m)}| \} = \frac{1}{\lambda^{\tau-s}} O \left( \|g\|_{**} + \frac{\|\phi\|_*}{(\lambda l)^{\frac{n}{2}}} \right).$$

An argument similar to the one used in (2.7) yields

$$\begin{aligned} \left| \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} \right| &\leq \frac{C}{\lambda^{\tau-s}} \left( \|g\|_{**} + \frac{\|\phi\|_*}{(\lambda l)^{\frac{n}{2}}} \right) \sum_{i=1}^{(m+1)^k} \frac{1}{(1 + |y - X^i|)^{n+2s}} \\ &\leq C \left( \|g\|_{**} + \frac{\|\phi\|_*}{(\lambda l)^{\frac{n}{2}}} \right) \gamma(y) \sum_{i=1}^{(m+1)^k} \frac{1}{(1 + |y - X^i|)^{\frac{n+2s}{2} + \tau}}. \end{aligned}$$

From the definition of the norm  $\|\cdot\|_{**}$ , we have

$$\left\| \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} \right\|_{**} \leq C \left( \|g\|_{**} + \frac{\|\phi\|_*}{(\lambda l)^{\frac{n}{2}}} \right).$$

Applying Lemma 2.1 to the first equation of the system (2.10), one get

$$\begin{aligned} &\left( \gamma(y) \sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}} \right)^{-1} |\phi(y)| \\ &\leq C \left( \|g\|_{**} + \left\| \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} \right\|_{**} + \left( \frac{1}{(\lambda l)^{\frac{4s}{n-2s}k}} + \frac{\sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau + \theta}}}{\sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}}} \right) \|\phi\|_* \right) \\ &\leq C \left( \|g\|_{**} + \frac{1}{(\lambda l)^{\frac{4s}{n-2s}k}} + \frac{\sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau + \theta}}}{\sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}}} \right). \end{aligned}$$

As a result, there exist a number  $i_0 \in \mathbb{N}$  and a large constant  $R > 0$ , such that

$$1 = \|\phi\|_* = \sup_{B_R(X^{i_0})} (\gamma(y) \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2} + \tau}})^{-1} |\phi(y)|. \quad (2.13)$$

Hence there is a constant  $c_0 > 0$  such that  $|\lambda^{\tau-s} \phi|_{L^\infty(B_R(X^{i_0}))} \geq c_0$ .

Applying Lemma A.8 to the equation (2.10), we know  $\lambda^{\tau-s} \phi$  is equi-continuous. Also  $\lambda^{\tau-s} |\phi(\cdot)|$  is uniformly bounded. In fact, we assume that  $y \in \Omega_1$  with no loss of generality. From the fact (2.12), we have

$$\lambda^{\tau-s} |\phi(y)| \leq \|\phi\|_* \sum_h \frac{1}{(1 + |y - X^h|)^{\frac{n}{2}}} \leq C + \sum_{h \neq 1} \frac{1}{|X^h - X^1|^{\frac{n}{2}}} \leq C.$$



Then the Arzelà-Ascoli Theorem yields that there is a function  $\tilde{\phi}$ , such that  $\lambda^{\tau-s}\phi(\cdot + P^{i_0})$  convergent to  $\tilde{\phi}$  uniformly on compact sets. Then

$$|\tilde{\phi}|_{L^\infty(B_{R+1}(0))} \geq c_0. \quad (2.14)$$

Using a similar argument as in [12, Lemma 7.3], we know  $\tilde{\phi}$  satisfies

$$\begin{cases} (-\Delta)^s \tilde{\phi} - \frac{n+2s}{n-2s} U_{0,\Lambda_{i_0}}^{\frac{4s}{n-2s}} \tilde{\phi} = 0, \\ \int_{\mathbb{R}^n} U_{0,\Lambda_{i_0}}^{\frac{4s}{n-2s}} \frac{\partial U_{0,\Lambda_{i_0}}}{\partial \Lambda_{i_0}} \tilde{\phi} = 0, \quad \int_{\mathbb{R}^n} U_{0,\Lambda_{i_0}}^{\frac{4s}{n-2s}} \frac{\partial U_{0,\Lambda_{i_0}}}{\partial P_j^{i_0}} \tilde{\phi} = 0, \quad j = 1, \dots, n. \end{cases}$$

Then  $\tilde{\phi} = 0$  by nondegeneracy, which is contradict to (2.14). Hence the solution  $\phi$  of the equation (2.10) satisfies  $\|\phi\|_* \leq C\|g\|_{**}$ .  $\square$

Combining Lemma 2.2, Lemma A.8 and the argument of [14, Proposition 4.1] (cf. [24, Proposition 2.2]), we have

**Proposition 2.3.** *For any  $g$  satisfying  $\|g\|_{**} < +\infty$ , (2.10) has a unique solution  $\phi = L_m(g) \in \dot{H}^s(\mathbb{R}^n) \cap C^{0,\alpha}(\mathbb{R}^n)$  with  $\alpha = \min\{2s, 1\}$ , such that  $\|L_m(g)\|_* \leq C\|g\|_{**}$ . The constant  $c_{ij}^{(m)}$  satisfies  $|c_{ij}^{(m)}| \leq \frac{C}{\lambda^{\tau-s}} \|g\|_{**}$ .*

Since we are interested in the solution of the form  $W_m + \phi_m$  of the equation (1.5), we now consider the following problem

$$\begin{cases} (-\Delta)^s \phi - \frac{n+2s}{n-2s} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}} \phi = N(\phi) + l_m + \sum_{i=1}^{(m+1)^k} \sum_{j=1}^{n+1} c_{ij}^{(m)} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j}, \\ \int_{\mathbb{R}^n} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} \phi dx = 0, \quad \phi \in \dot{H}^s(\mathbb{R}^n), \quad i = 1, \dots, (m+1)^k, \quad j = 1, \dots, n+1, \end{cases} \quad (2.15)$$

where

$$N(\phi) = K\left(\frac{x}{\lambda}\right) \left( (W_m + \phi)_+^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}} - \frac{n+2s}{n-2s} W_m^{\frac{4s}{n-2s}} \phi \right)$$

and

$$l_m = K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} - \sum_{i=1}^{(m+1)^k} U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}}.$$

**Lemma 2.4.** *For the terms  $N(\phi)$  and  $l_m$  defined above, we have the following estimates*

$$\|N(\phi)\|_{**} \leq C\|\phi\|_*^{\min\{2, 2^*(s)-1\}},$$

$$\|l_m\|_{**} \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}}.$$

*Proof.* The proof of the first estimation is rather standard (cf. [29, Lemma 2.4] for ideas). We only prove the second estimate.

Without loss of generality, we assume  $x \in \Omega_1$ . Then

$$\begin{aligned} l_m &= K\left(\frac{x}{\lambda}\right)W_m^{\frac{n+2s}{n-2s}} - U_{P^1, \Lambda_1}^{\frac{n+2s}{n-2s}} - \sum_{h \neq 1} U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \\ &= \left(K\left(\frac{x}{\lambda}\right) - 1\right) U_{P^1, \Lambda_1}^{\frac{n+2s}{n-2s}} + O\left(\left(\sum_{h \neq 1} U_{P^h, \Lambda_h}\right)^{\frac{n+2s}{n-2s}} + U_{P^1, \Lambda_1}^{\frac{4s}{n-2s}} \sum_{h \neq 1} U_{P^h, \Lambda_h}\right). \end{aligned} \quad (2.16)$$

The two error terms in (2.16) can be estimated by using Lemma A.3 under different cases.

**Case 1:**  $x \in \Omega_1 \cap B_1^c \cap B_{1,m}$ , we have  $\gamma(x) = 1$ . Using Lemma A.3, we have

$$\begin{aligned} \left(\sum_{h \neq 1} U_{P^h, \Lambda_h}\right)^{\frac{n+2s}{n-2s}} &\leq \frac{C}{(\lambda l)^{\frac{n+2s}{n-2s}k}} \frac{1}{(1 + |x - X^1|)^{n+2s - \frac{n+2s}{n-2s}k}} \\ &\leq \frac{C}{(\lambda l)^{\frac{n+2s}{2} - \tau + k}} \frac{1}{(1 + |x - X^1|)^{\frac{n+2s}{2} + \tau - k}} \\ &\leq \frac{C}{(\lambda l)^{\frac{n+2s}{2} - \tau}} \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{2} + \tau}}. \end{aligned}$$

**Case 2:**  $x \in \Omega_1 \cap B_1$ , it holds that  $|x - X^i| \geq \frac{1}{2}|X^i - X^1| \geq \frac{1}{2}\lambda l$  for  $i \neq 1$ . From Lemma A.3,

$$\left(\sum_{h \neq 1} U_{P^h, \Lambda_h}\right)^{\frac{n+2s}{n-2s}} \leq \begin{cases} \frac{C}{(\lambda l)^{\frac{n+2s}{2} - \tau}} \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{2} + \tau}}, & \text{if } x \in \Omega_1 \cap B_1 \cap B_\lambda^c(X_1), \\ C \left(\frac{1 + |x - X^1|}{\lambda}\right)^{\tau - s} \frac{\lambda^{\tau - s}}{(\lambda l)^{\frac{n}{2}}} \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{2} + \tau}}, & \text{if } x \in \Omega_1 \cap B_1 \cap B_\lambda(X^1). \end{cases}$$

**Case 3:**  $x \in \Omega_1 \cap B_{1,m}^c$ , we can get

$$\left(\sum_{h \neq 1} U_{P^h, \Lambda_h}\right)^{\frac{n+2s}{n-2s}} \leq \frac{C m^{\frac{n+2s}{n-2s}k}}{(1 + |x - X^1|)^{n+2s}} \leq \frac{C m^{\frac{4s}{n-2s}k}}{(1 + |x - X^1|)^{2s}} \frac{(1 + C[\frac{m}{2}])^k}{(1 + |x - X^1|)^n}.$$

Following the proof of Lemma A.3, we have

$$\begin{aligned} \sum_h \frac{1}{(1 + |x - X^h|)^n} &\geq \frac{C}{(1 + |x - X^1|)^n} \left(1 + 2^{-k} \int_{[0, [\frac{m}{2}] + 1]^k \setminus [0, 1]^k} \frac{1}{(1 + \frac{\lambda l}{1 + |x - X^1|} |z|)^n} dz\right) \\ &\geq \frac{(1 + C[\frac{m}{2}])^k}{(1 + |x - X^1|)^n}. \end{aligned}$$

Since in the domain  $\Omega_1 \cap B_{1,m}^c$ , we can get  $|x - X^h| \geq \frac{1}{2}\lambda l$  for  $h = 1, \dots, (m+1)^k$ , then

$$\left(\sum_{h \neq 1} U_{P^h, \Lambda_h}\right)^{\frac{n+2s}{n-2s}} \leq \frac{C}{(\lambda l)^{2s}} \sum_h \frac{1}{(1 + |x - X^h|)^n} \leq \frac{C}{(\lambda l)^{\frac{n+2s}{2} - \tau}} \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{2} + \tau}}.$$

Combining these three cases above, we have  $\|(\sum_{h \neq 1} U_{P^h, \Lambda_h})^{\frac{n+2s}{n-2s}}\|_{**} \leq \frac{C}{(\lambda l)^{\frac{n+2s}{2}-\tau}}$ . By the same procedure, we can also get the estimation  $\|U_{P^1, \Lambda_1}^{\frac{4s}{n-2s}} \sum_{h \neq 1} U_{P^h, \Lambda_h}\|_{**} \leq \frac{C}{(\lambda l)^{\frac{n+2s}{2}-\tau}}$ .

At last, we estimate the first term in (2.16).

In the case of  $|x - X^1| \geq \lambda$ , we have  $\gamma(x) = 1$ . Then

$$\left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{P^1, \Lambda_1}^{\frac{n+2s}{n-2s}} \leq \frac{C}{(1 + |x - X^1|)^{n+2s}} \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}} \gamma(x) \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{2}+\tau}}. \quad (2.17)$$

In the case of  $|x - X^1| < \lambda$ , it holds  $\frac{1+|x-X^1|}{\lambda} \leq C$ . The condition  $(H_3)$  yields

$$\begin{aligned} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{P^1, \Lambda_1}^{\frac{n+2s}{n-2s}} &\leq C \frac{|x - X^1|^\beta}{\lambda^\beta} \frac{1}{(1 + |x - X^1|)^{n+2s}} \\ &\leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}} \left( \frac{1 + |x - X^1|}{\lambda} \right)^{\tau-s} \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{2}+\tau}} \\ &\leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}} \gamma(y) \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{2}+\tau}}. \end{aligned} \quad (2.18)$$

Summarizing (2.17) and (2.18), we have

$$\left\| \left( K\left(\frac{x}{\lambda}\right) - 1 \right) U_{P^1, \Lambda_1}^{\frac{n+2s}{n-2s}} \right\|_{**} \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}}.$$

Hence this lemma follows.  $\square$

**Proposition 2.5.** *For  $\lambda$  large enough, the problem (2.15) has a unique solution  $\phi_m \in \dot{H}^s(\mathbb{R}^n) \cap C^{0,\alpha}(\mathbb{R}^n)$  with  $\alpha = \min\{2s, 1\}$ , such that  $\|\phi_m\|_* \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}}$ . The constants  $c_{ij}^{(m)}$  satisfy  $|c_{ij}^{(m)}| \leq C\lambda^{-\frac{n}{2}}$ .*

*Proof.* We define

$$E = \left\{ \varphi \in \dot{H}(\mathbb{R}^n) \cap C(\mathbb{R}^n) : \|\varphi\|_* \leq \frac{1}{\lambda^{\frac{n+2s}{2}-\tau-\epsilon_1}}, \int_{\mathbb{R}^n} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,j} \varphi = 0, \begin{matrix} i = 1, \dots, (m+1)^k, \\ j = 1, \dots, n+1 \end{matrix} \right\},$$

where  $\epsilon_1 = \min\{\frac{1}{4}, \frac{2s}{n+2s}\}(\frac{n+2s}{2} - \tau)$ . Notice that  $(E, \|\cdot\|_*)$  is a metric space.

In order to use the contraction map theorem, we define  $A\varphi := L_m(N(\varphi) + l_m)$ , where  $L_m$  is an operator defined in Proposition 2.3.

Firstly, we show that  $A$  maps  $E$  into itself for  $\lambda$  large. Combining Proposition 2.3 and Lemma 2.4, we have  $\forall \varphi \in E$ ,

$$\|A\varphi\|_* \leq C(\|N(\varphi)\|_{**} + \|l_m\|_{**}) \leq C(\|\varphi\|_*^{\min\{2, 2^*(s)-1\}} + \|l_m\|_{**}) \leq \frac{1}{\lambda^{\frac{n+2s}{2}-\tau-\epsilon_1}}.$$

Secondly, we prove  $A$  is an contraction map for  $\lambda$  large.

Choose  $\varphi_1, \varphi_2 \in E$  with  $\varphi_1 \neq \varphi_2$ . If  $N \geq 6s$ , we have

$$\begin{aligned}
|N(\varphi_1) - N(\varphi_2)| &= |N'(t\varphi_1 + (1-t)\varphi_2)(\varphi_1 - \varphi_2)| \\
&\leq C(|\varphi_1|^{\frac{4s}{n-2s}} + |\varphi_2|^{\frac{4s}{n-2s}})(|\varphi_1 - \varphi_2|) \\
&\leq C(\|\varphi_1\|_*^{\frac{4s}{n-2s}} + \|\varphi_2\|_*^{\frac{4s}{n-2s}})\|\varphi_1 - \varphi_2\|_* \left( \gamma(x) \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |x - X^h|)^{\frac{n-2s}{2} + \tau}} \right)^{\frac{n+2s}{n-2s}} \\
&\leq C(\|\varphi_1\|_*^{\frac{4s}{n-2s}} + \|\varphi_2\|_*^{\frac{4s}{n-2s}})\|\varphi_1 - \varphi_2\|_* \left( \gamma(x) \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |x - X^h|)^{\frac{n+2s}{2} + \tau}} \right).
\end{aligned}$$

We remind that in the last inequality, we have used the Hölder inequality. Hence  $\|N(\varphi_1) - N(\varphi_2)\|_{**} \leq C(\|\varphi_1\|_*^{\frac{4s}{n-2s}} + \|\varphi_2\|_*^{\frac{4s}{n-2s}})\|\varphi_1 - \varphi_2\|_*$ .

In the case of  $N \leq 6s$ , we also have  $\|N(\varphi_1) - N(\varphi_2)\|_{**} \leq C(\|\varphi_1\|_*^{\min\{\frac{4s}{n-2s}, 1\}} + \|\varphi_2\|_*^{\min\{\frac{4s}{n-2s}, 1\}})\|\varphi_1 - \varphi_2\|_*$  by a similar argument.

Then there hold

$$\begin{aligned}
\|A\varphi_1 - A\varphi_2\|_* &\leq C\|N(\varphi_1) - N(\varphi_2)\|_{**} \\
&\leq C(\|\varphi_1\|_*^{\min\{\frac{4s}{n-2s}, 1\}} + \|\varphi_2\|_*^{\min\{\frac{4s}{n-2s}, 1\}})\|\varphi_1 - \varphi_2\|_* \\
&\leq \frac{C}{\lambda^{(\frac{n+2s}{2} - \tau - \epsilon_1) \min\{\frac{4s}{n-2s}, 1\}}} \|\varphi_1 - \varphi_2\|_*.
\end{aligned}$$

For  $\lambda$  large enough, we get  $\|A\varphi_1 - A\varphi_2\|_* \leq \frac{1}{2}\|\varphi_1 - \varphi_2\|_*$ .

Using the contracting map theorem, we know that there is a unique  $\phi_m \in E$ , such that  $A(\phi_m) = \phi_m$ , i.e.  $\phi_m$  is a unique solution of (2.15) in  $E$ . From Proposition 2.3 and Lemma 2.4, we know  $\phi_m \in \dot{H}^s(\mathbb{R}^n) \cap C^{0,\alpha}(\mathbb{R}^n)$  satisfying  $\|\phi_m\|_* = \|A\phi_m\|_* \leq \frac{C}{\lambda^{\frac{n+2s}{2} - \tau}}$  and  $|c_{ij}^{(m)}| \leq C\lambda^{-\frac{n}{2}}$ , since  $(\frac{n+2s}{2} - \tau - \epsilon_1) \min\{2, 2^*(s) - 1\} > \frac{n+2s}{2} - \tau$ .

□

### 3. PROOF OF THE MAIN THEOREM

Let  $\Lambda := (\Lambda_1, \dots, \Lambda_{(m+1)^k}) \in \mathbb{R}_+^{(m+1)^k}$  and  $P := (P^1, \dots, P^{(m+1)^k}) \in \mathbb{R}^{n \times (m+1)^k}$ , in which  $P^i = (P_1^i, \dots, P_n^i) \in \mathbb{R}^n$  for  $i = 1, 2, \dots, (m+1)^k$ . We define  $J(P, \Lambda) = I(W_m + \phi_m)$ , where  $\phi_m$  is a unique small solution obtained by Proposition 2.5. A standard argument shows that from a critical point of  $J$ , we can get a critical point of  $I$  of the form  $W_m + \phi_m$  (for example, cf. [14, Lemma 6.1] for ideas).

**Proposition 3.1.** *For  $\lambda$  large, we have the following expansions*

$$\frac{\partial J}{\partial P_j^i}(P, \Lambda) = -\frac{c_3 a_j}{\Lambda_i^{\beta-2} \lambda^\beta} (P_j^i - X_j^i) + O\left(\frac{|P^i - X^i|^2}{\lambda^\beta}\right) + o(\lambda^{-\beta}), \quad (3.1)$$

and

$$\frac{\partial J}{\partial \Lambda_i}(P, \Lambda) = -\frac{c_1}{\Lambda_i^{\beta+1} \lambda^\beta} + \sum_{h \neq i} \frac{c_2}{\Lambda_i(\Lambda_i \Lambda_h)^{\frac{n-2s}{2}} |X^i - X^h|^{n-2s}} + O\left(\frac{|P^i - X^i|^{\min\{2, \beta-1\}}}{\lambda^\beta}\right) + o(\lambda^{-\beta}), \quad (3.2)$$

where  $i = 1, \dots, (m+1)^k$  and  $j = 1, \dots, n$  and the constant  $c_1, c_2, c_3$  are positive.

*Proof.* A simple calculation yields

$$\begin{aligned} \frac{\partial J}{\partial P_j^i}(P, \Lambda) &= \langle I'(W_m + \phi_m), \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} + \frac{\partial \phi_m}{\partial P_j^i} \rangle \\ &= \frac{\partial I}{\partial P_j^i}(W_m) + \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) (W_m^{\frac{n+2s}{n-2s}} - (W_m + \phi_m)_+^{\frac{n+2s}{n-2s}}) \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \\ &\quad + \sum_{t=1}^{(m+1)^k} \sum_{h=1}^{n+1} c_{th}^{(m)} \int_{\mathbb{R}^n} U_{P^t, \Lambda_t}^{\frac{4s}{n-2s}} Z_{t,h} \frac{\partial \phi_m}{\partial P_j^i}. \end{aligned}$$

The functional  $\frac{\partial I}{\partial P_j^i}(W_m)$  is expanded in the Proposition B.6. So we only need to estimate the last two terms in the equality above.

From Lemma A.4, we see that for  $\lambda$  large enough,  $\{x : W_m \leq -\phi_m\} \subset \{x : \frac{1}{2}W_m \leq |\phi_m|\} \subset \cup_h(\Omega_h \cap B_h^c)$ . Then we have

$$\begin{aligned} &\int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) ((W_m + \phi_m)_+^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}}) \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} dx \\ &= \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) ((W_m + \phi_m)^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}}) \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} dx - \int_{-\phi_m \geq W_m} K\left(\frac{x}{\lambda}\right) (W_m + \phi_m)^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} dx \\ &= \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}} \phi_m \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} dx + O\left(\int_{|\phi_m| \geq \frac{1}{2}W_m} |\phi_m|^{\frac{n+2s}{n-2s}} U_{P^i, \Lambda_i} dx\right. \\ &\quad \left.+ \int_{|\phi_m| < \frac{1}{2}W_m} W_m^{\frac{6s-n}{n-2s}} \phi_m^2 \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} dx\right) \\ &= \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}} \phi_m \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} dx + O\left(\int_{\cup_h(\Omega_h \cap B_h^c)} |\phi_m|^{\frac{n+2s}{n-2s}} U_{P^i, \Lambda_i} dx\right. \\ &\quad \left.+ \int_{\mathbb{R}^n} W_m^{\frac{6s-n}{n-2s}} \phi_m^2 U_{P^i, \Lambda_i} dx\right). \end{aligned}$$

Using Proposition 2.5, Lemma A.6 and Lemma A.7, we have

$$\int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) ((W_m + \phi_m)_+^{\frac{n+2s}{n-2s}} - W_m^{\frac{n+2s}{n-2s}}) \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} dx = O(\lambda^{-n}) = o(\lambda^{-\beta}).$$

By using the orthogonal condition of (2.15) and Lemma A.1, we have

$$\begin{aligned} &\left| \sum_{t=1}^{(m+1)^k} \sum_{h=1}^{n+1} c_{th} \int_{\mathbb{R}^n} U_{P^t, \Lambda_t}^{\frac{4s}{n-2s}} Z_{t,h} \frac{\partial \phi_m}{\partial P_j^i} dx \right| = \left| \sum_{h=1}^{n+1} c_{ih} \int_{\mathbb{R}^n} \frac{\partial}{\partial P_j^i} (U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} Z_{i,h}) \phi_m dx \right| \\ &\leq \frac{C}{\lambda^{\frac{n}{2}}} \frac{\|\phi_m\|_*}{\lambda^{\tau-s}} \int_{\mathbb{R}^n} \frac{1}{(1+|x-X^i|)^{n+2s}} \sum_{r=1}^{(m+1)^k} \frac{1}{(1+|x-X^r|)^{\frac{n}{2}}} dx \leq C\lambda^{-n}. \end{aligned}$$

Hence we can get (3.1). The estimation (3.2) can be derived by the same procedure along with Proposition B.5.

□

**Remark 3.2.** From Proposition 3.1, we know that there exist bounded functions  $\Xi_j^i = \Xi_j^i(P, \Lambda, \lambda)$  and  $\Theta_j^i = \Theta_j^i(P, \Lambda, \lambda)$ ,  $i = 1, \dots, (m+1)^k$ ;  $j = 1, \dots, n+1$  satisfying

$$|\Xi_j^i| \leq C, \text{ where } C \text{ is constant independent of } m$$

and

$$|\Theta_j^i| \leq C_\lambda, \text{ where } C_\lambda \text{ is a constant only depend on } \lambda, \text{ and } C_\lambda \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

such that

$$\frac{\partial J}{\partial P_j^i}(P, \Lambda) = -\frac{c_3 a_j}{\Lambda_i^{\beta-2} \lambda^\beta} (P_j^i - X_j^i) + \frac{|P^i - X^i|^2}{\lambda^\beta} \Xi_j^i + \lambda^{-\beta} \Theta_j^i,$$

and

$$\frac{\partial J}{\partial \Lambda_i}(P, \Lambda) = -\frac{c_1}{\Lambda_i^{\beta+1} \lambda^\beta} + \sum_{h \neq i} \frac{c_2}{\Lambda_i(\Lambda_i \Lambda_h)^{\frac{n-2s}{2}} |X^i - X^h|^{n-2s}} - \frac{|P^i - X^i|^{\min\{2, \beta-1\}}}{\lambda^\beta} \Xi_{n+1}^i - \lambda^{-\beta} \Theta_{n+1}^i.$$

*Proof of Theorem 1.2.* Firstly we look for the solution of (1.5) of the form  $W_m + \phi_m$ ,  $m < \infty$ .

$$\text{It is equivalent to solving the system } \begin{cases} \frac{\partial J}{\partial P_j^i}(P, \Lambda) = 0, \\ \frac{\partial J}{\partial \Lambda_i}(P, \Lambda) = 0, \end{cases} \quad i = 1, 2, \dots, (m+1)^k; \quad j = 1, 2, \dots, n,$$

that is

$$\begin{cases} \frac{c_3 a_j}{\Lambda_i^{\beta-2} \lambda^\beta} (P_j^i - X_j^i) = \frac{|P^i - X^i|^2}{\lambda^\beta} \Xi_j^i + \lambda^{-\beta} \Theta_j^i; \\ -\frac{c_1}{\Lambda_i^{\beta+1} \lambda^\beta} + \sum_{h \neq i} \frac{c_2}{\Lambda_i(\Lambda_i \Lambda_h)^{\frac{n-2s}{2}} |X^i - X^h|^{n-2s}} = \frac{|P^i - X^i|^{\min\{2, \beta-1\}}}{\lambda^\beta} \Xi_{n+1}^i + \lambda^{-\beta} \Theta_{n+1}^i. \end{cases} \quad (3.3)$$

$$\text{To simplify the equations (3.3), we denote } d_j = \Lambda_j^{-\frac{n-2s}{2}} \text{ and } A_{ih} = \begin{cases} 0, & \text{if } i = h, \\ \frac{(\lambda l)^{n-2s}}{|X^i - X^h|^{n-2s}}, & \text{if } i \neq h. \end{cases}$$

The equations (3.3) can be written as

$$\begin{cases} P_j^i - X_j^i = \frac{\Lambda_i^{\beta-2} \Xi_j^i}{c_3 a_j} |P^i - X^i|^2 + \frac{\Lambda_i^{\beta-2}}{c_3 a_j} \Theta_j^i, \\ c_2 \sum_{h \neq i} A_{ih} d_h - c_1 d_i^{\frac{2\beta}{n-2s}-1} = \Lambda_i^{\frac{n-2s}{2}+1} \Xi_{n+1}^i |P^i - X^i|^{\min\{2, \beta-1\}} + \Lambda_i^{\frac{n-2s}{2}+1} \Theta_{n+1}^i, \end{cases} \quad (3.4)$$

where  $i = 1, 2, \dots, (m+1)^k$  and  $j = 1, 2, \dots, n$ .

Define a function  $F(z) := \frac{c_2}{2} \sum_{h \neq i} A_{ih} z_i z_h - \frac{(n-2s)c_1}{2\beta} \sum_h z_h^{\frac{2\beta}{n-2s}}$ , where  $z = (z_1, z_2, \dots, z_{(m+1)^k}) \in \mathbb{R}^{(m+1)^k}$ . Obviously,  $F(z)$  has a maximum point  $b = (b_1, b_2, \dots, b_{(m+1)^k}) \in \mathbb{R}_+^{(m+1)^k}$ . It holds that

$$c_2 \sum_{h \neq i} A_{ih} b_h - c_1 b_i^{\frac{2\beta}{n-2s}-1} = 0, \quad i = 1, \dots, (m+1)^k. \quad (3.5)$$

**Claim:** Each component  $b_i$  of  $b$  satisfies  $0 < C'_1 \leq b_i \leq C'_2$  for some constant  $C'_1$  and  $C'_2$ .

Suppose that  $b_1 \leq b_i \leq b_2$ . Using the definition of  $A_{ih}$ , we know  $\sum_{h \neq i} A_{ih}$  is bounded. From (3.5), we can get

$$c_1 b_2^{\frac{2\beta}{n-2s}-1} = c_2 \sum_{h \neq 2} A_{2h} b_h \leq C_3 b_2,$$

which tell us  $b_2$  is bounded from above.

Using (3.5) again, we have

$$c_1 b_1^{\frac{2\beta}{n-2s}-1} = c_2 \sum_{h \neq 1} A_{1h} b_h \geq c_2 \sum_{h \neq 1} A_{1h} b_1 \geq c_2 A_{12} b_1,$$

which implies  $b_1$  is bounded from below, away from zero. Hence the Claim follows.

We can choose a small  $\delta_0 > 0$  such that  $b_2^{-\frac{2}{n-2s}} - \delta_0 > 0$ . The constant  $C_1$  and  $C_2$  in the introduction can be defined by

$$C_1 = b_2^{-\frac{2}{n-2s}} - \delta_0 \quad \text{and} \quad C_2 = b_1^{-\frac{2}{n-2s}} + \delta_0. \quad (3.6)$$

For any  $x = (x_1, \dots, x_{(m+1)^k}) \in \mathbb{R}^{(m+1)^k}$ , we denote  $\|x\|_0 = \max_j \{|x_j|\}$ . Let  $|\frac{x_{i_0}}{b_{i_0}}| = \|\frac{x}{b}\|_0$ . From the claim above, we know  $|x_{i_0}| \geq C\|x\|_0$ . Using (3.5), we have

$$\begin{aligned} |(D^2 F(b)x)_{i_0}| &= |c_2 \sum_{h \neq i_0} A_{i_0 h} x_h - c_1 (\frac{2\beta}{n-2s} - 1) b_{i_0}^{\frac{2\beta}{n-2s}-2} x_{i_0}| \\ &\geq c_1 (\frac{2\beta}{n-2s} - 1) b_{i_0}^{\frac{2\beta}{n-2s}-2} |x_{i_0}| - c_2 |\sum_{h \neq i} A_{i_0 h} x_h| \\ &\geq c_1 (\frac{2\beta}{n-2s} - 1) b_{i_0}^{\frac{2\beta}{n-2s}-1} |\frac{x_{i_0}}{b_{i_0}}| - c_2 \sum_{h \neq i} A_{i_0 h} b_h |\frac{x_{i_0}}{b_{i_0}}| \\ &= c_1 (\frac{2\beta}{n-2s} - 2) b_{i_0}^{\frac{2\beta}{n-2s}-2} |x_{i_0}| \geq C_4 \|x\|_0. \end{aligned}$$

From the definition of  $\|\cdot\|_0$ , we get  $\|D^2 F(b)x\|_0 \geq C_4 \|x\|_0$ .

Let  $\theta = (\theta_1, \dots, \theta_{(m+1)^k}) \in \mathbb{R}^{(m+1)^k}$  whose component  $\theta_i := d_i - b_i$ ,  $i = 1, \dots, (m+1)^k$ . We define  $X := (X^1, \dots, X^{(m+1)^k}) \in \mathbb{R}^{n \times (m+1)^k}$ , in which  $X^i = (X_1^i, \dots, X_n^i) \in \mathbb{R}^n$  for  $i = 1, \dots, (m+1)^k$ . For any  $Y = (Y^1, \dots, Y^{(m+1)^k}) \in \mathbb{R}^{n \times (m+1)^k}$ , we use the notation  $\|Y\| := \max_{i=1, \dots, (m+1)^k} \{|Y^i|\}$  to denote the maximum norm.

To simplify the equations (3.4), we need to define some vector value functions below. Let  $\Xi^{(1)} := \Xi^{(1)}(P, \Lambda, \lambda) \in \mathbb{R}^{n \times (m+1)^k}$  and  $\Theta^{(1)} := \Theta^{(1)}(P, \Lambda, \lambda) \in \mathbb{R}^{n \times (m+1)^k}$  with their exponents defined by

$$(\Xi^{(1)})_j^i = \frac{\Lambda_i^{\beta-2} \Xi_j^i |P^i - X^i|^2}{c_3 a_j \|P - X\|^2} \quad \text{and} \quad (\Theta^{(1)})_j^i = \frac{\Lambda_i^{\beta-2}}{c_3 a_j} \Theta_j^i, \quad i = 1, \dots, (m+1)^k; j = 1, \dots, n.$$

Let  $\Xi^{(2)} := \Xi^{(2)}(P, \Lambda, \lambda) \in \mathbb{R}^{(m+1)^k}$ ,  $\Theta^{(2)} := \Theta^{(2)}(P, \Lambda, \lambda) \in \mathbb{R}^{(m+1)^k}$  with exponents defined by

$$(\Xi^{(2)})^i = \Lambda_i^{\frac{n-2s}{2}+1} \Xi_{n+1}^i \frac{|P^i - X^i|^{\min\{2, \beta-1\}}}{\|P - X\|^{\min\{2, \beta-1\}}} \quad \text{and} \quad (\Theta^{(2)})^i = \Lambda_i^{\frac{n-2s}{2}+1} \Theta_{n+1}^i, \quad i = 1, \dots, (m+1)^k.$$

Define  $\Pi(\theta) := (\Pi(\theta)^1, \dots, \Pi(\theta)^{(m+1)^k})$ , where  $\Pi(\theta)^i$  ( $i = 1, \dots, (m+1)^k$ ) is defined by

$$(\Pi(\theta))^i = \int_0^1 (D^3 F(b + s\theta)\theta)_i (1-s) ds = \int_0^1 c_1 (\frac{2\beta}{n-2s} - 1) (\frac{2\beta}{n-2s} - 2) (b_i + s\theta_i)^{\frac{2\beta}{n-2s}-3} \theta_i^2.$$

From their definition, we know there is a constant  $C$  and a constant  $C_\lambda$  satisfying  $C_\lambda \rightarrow 0$  as  $\lambda \rightarrow \infty$  such that  $\|\Xi^{(1)}\| \leq C$ ,  $\|\Xi^{(2)}\|_0 \leq C$ ;  $\|\Theta^{(1)}\| \leq C_\lambda$  and  $\|\Theta^{(2)}\|_0 \leq C_\lambda$ .

Using these notations and Taylor expansion, we can write the equations (3.4) into another form:

$$\begin{cases} P - X = \|P - X\|^2 \Xi^{(1)} + \Theta^{(1)}; \\ D^2 F(b) \theta = \|P - X\|^{\min\{2, \beta-1\}} \Xi^{(2)} + \Theta^{(2)} + \Pi(\theta), \end{cases} \quad (3.7)$$

Let

$$B = \left( \prod_{i=1}^{(m+1)^k} B_{2C_\lambda}(X^i) \right) \times B_{3C_4^{-1}C_\lambda}(0) \in \mathbb{R}^{n \times (m+1)^k} \times \mathbb{R}^{(m+1)^k}.$$

Define a function

$$\begin{aligned} G : B &\rightarrow B \\ (P, \theta) &\mapsto (X + \Xi^{(1)} \|P - X\|^2 + \Theta^{(1)}, D^2 F(b)^{-1} (\|P - X\|^{\beta-1} \Xi^{(2)} + \Theta^{(2)} + \Pi(\theta))) \end{aligned}$$

For each  $(P, \theta) \in B$ , Choose  $C_\lambda$  small enough, we have

$$\|\Xi^{(1)} \|P - X\|^2 + \Theta^{(1)}\| \leq C(2C_\lambda)^2 + C_\lambda \leq 2C_\lambda^2,$$

and

$$\|D^2 F(b)^{-1} (\|P - X\|^{\min\{2, \beta-1\}} \Xi^{(2)} + \Pi(\theta))\|_0 \leq C_4^{-1} (C C_\lambda^{\min\{2, \beta-1\}} + C_\lambda + C\theta^2) \leq 3C_4^{-1} C_\lambda.$$

Since  $C_\lambda \rightarrow 0$  as  $\lambda \rightarrow \infty$ , so for  $\lambda$  large enough, we use the Brouwer fixed-point theorem to get a solution  $(P^1, \dots, P^{(m+1)^k}, \theta)$  of (3.7) in  $B$ . It holds that

$$|P^i - X^i| \leq 2C_\lambda \quad \text{and} \quad |\theta_i| = |b_i - \Lambda_i^{-\frac{n-2s}{2}}| \leq 3C_4^{-1} C_\lambda.$$

Hence we find a critical point of  $I$  of the form  $u_m := W_m + \phi_m$  with  $m < \infty$ .

Next, we prove  $u_m$  is a positive function. Denote  $u_m^- = \min\{0, u_m\}$  and  $u_m^+ = u_m - u_m^-$ . Then we have

$$\int_{\mathbb{R}^n} (-\Delta)^s u_m(x) u_m^-(x) dx = \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) u_m^{\frac{n+2s}{n-2s}}(x) u_m^-(x) dx.$$

From the definition of  $(-\Delta)^s$ ,

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^s u_m u_m^- dx &= \int_{\mathbb{R}^n} (-\Delta)^s u_m^- u_m^- dx + \int_{\mathbb{R}^n} (-\Delta)^s u_m^+ u_m^- dx \\ &= \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u_m^-|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_m^+(x) - u_m^+(y)) u_m^-(x)}{|x - y|^{n+2s}} dx dy \\ &\geq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u_m^-|^2 dx. \end{aligned}$$

The Hardy-Littlewood-Sobolev inequality yields

$$\begin{aligned} \left( \int_{\mathbb{R}^n} |u_m^-|^{\frac{2n}{n-2s}} dx \right)^{\frac{n-2s}{n}} &\leq C \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u_m^-|^2 dx \\ &\leq C \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) u_m^{\frac{n+2s}{n-2s}}(x) u_m^-(x) dx \leq C \int_{\mathbb{R}^n} |u_m^-|^{\frac{2n}{n-2s}} dx. \end{aligned}$$



Suppose  $u_m^- \not\equiv 0$ , we have  $\int_{\mathbb{R}^n} |u_m^-|^{\frac{2n}{n-2s}} dx \geq C$ . It is easy to get  $u_m^- \leq |\phi_m|$ . From this fact,

$$\begin{aligned} \int_{\mathbb{R}^n} |u_m^-|^{\frac{2n}{n-2s}} dx &\leq \int_{\mathbb{R}^n} |\phi_m|^{\frac{2n}{n-2s}} dx \\ &\leq \|\phi_m\|_*^{\frac{2n}{n-2s}} \int_{\mathbb{R}^n} \left( \gamma(x) \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}} \right)^{\frac{2n}{n-2s}} dx \\ &\leq \|\phi_m\|_*^{\frac{2n}{n-2s}} C(m, k) \sum_h \int_{\mathbb{R}^n} \frac{1}{(1+|x-X^h|)^{n+\frac{2n}{n-2s}\tau}} dx. \end{aligned}$$

So we get  $C \leq \int_{\mathbb{R}^n} |u_m^-|^{\frac{2n}{n-2s}} dx \leq C(m, k) \|\phi_m\|_* \rightarrow 0$  as  $\lambda \rightarrow \infty$ , which is impossible. Hence  $u_m \geq 0$ . Suppose there is a point  $x_0$  such that  $u_m(x_0) = 0$ , then

$$0 = K\left(\frac{x_0}{\lambda}\right) u_m^{\frac{n+2s}{n-2s}}(x_0) = (-\Delta)^s u_m(x_0) = P.V. \int_{\mathbb{R}^n} \frac{u_m(x_0) - u_m(y)}{|x_0 - y|^{n+2s}} dy = P.V. \int_{\mathbb{R}^n} \frac{-u_m(y)}{|x_0 - y|^{n+2s}} dy.$$

Then  $u_m \equiv 0$  which is impossible. Hence  $u_m > 0$ .

According to Proposition 2.5,  $u_m = W_m + \phi_m \in C^{0,\alpha}(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$ . Using local Schauder estimate [18, Proposition 2.11] and a bootstrap argument, we know  $u_m \in C_{loc}^{2,\alpha'}(\mathbb{R}^n)$ , for some  $\alpha' \in (0, 1)$ .

What is more,  $|u_m|_{L^\infty(\mathbb{R}^n)} \leq C$  with  $C$  independent of  $m$ . In fact, Choosing  $x \in \Omega_1$  with no loss of generality, we have

$$W_m(x) \leq \sum_{i=1}^{(m+1)^k} \frac{C}{(1+|x-X^i|)^{n-2s}} \leq C + \sum_{i \neq 1}^{(m+1)^k} \frac{C}{|X^i - X^1|^{n-2s}} \leq C + \frac{C}{(\lambda l)^{n-2s}} \leq C,$$

and

$$|\phi_m| \leq \|\phi_m\|_* \sum_{h=1}^{(m+1)^k} \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}} \leq \|\phi_m\|_* \sum_{h=1}^{\infty} \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}} \leq C \|\phi_m\|_* \leq C.$$

Since  $\phi_m$  satisfies the equation  $(-\Delta)^s \phi_m - \frac{n+2s}{n-2s} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}} \phi_m = N(\phi_m) + l_m$ , then from Lemma A.8, Lemma 2.4 and Proposition 2.5, we know that for any  $x, y \in \mathbb{R}^n$  with  $x \neq y$ , there holds

$$\frac{|\phi_m(x) - \phi_m(y)|}{|x - y|^\alpha} \leq \frac{C}{\lambda^{\tau-s}} \max\{\|\phi_m\|_*, \|N(\phi_m)\|_{**} + \|l_m\|_{**}\} \leq C, \text{ where } \alpha = \min\{1, 2s\}.$$

Also from simple calculation, we get for any  $x \in \mathbb{R}^n$  and  $R > 0$ ,  $\|W_m\|_{C^{0,\alpha}(B_{2R}(x))} \leq C(n, R)$ , where  $C(n, R)$  is a constant independent of  $m$ . Hence  $\|u_m\|_{C^{0,\alpha}(B_{2R}(x))} \leq C(n, R)$ . Local Schauder estimate and a bootstrap argument yields that  $\|u_m\|_{C^{2,\alpha'}(B_R(x))} \leq C(n, R)$ . Thanks to Azellà-Ascoli theorem, we have  $u_m$  convergent uniformly to a  $C_{loc}^{2,\alpha'}$  function  $u_\infty = W_\infty + \phi_\infty$  on compact sets as  $m \rightarrow \infty$ . We know  $u_\infty$  satisfies  $|u_\infty|_{L^\infty(\mathbb{R}^n)} \leq C$  and  $\|u_\infty\|_{C^{2,\alpha'}(B_R(x))} \leq C(n, R)$ .

We will show that  $u_\infty$  satisfies the equation (1.5). Let  $v_m = u_m - u_\infty$ . From above, we know  $v_m$  has the property  $|v_m|_{L^\infty(\mathbb{R}^n)} \leq C$ ;  $|v_m|_{C^2(B_1(x))} \leq C$  and  $v_m \rightarrow 0$  uniformly on compact sets.

From the definition of  $(-\Delta)^s$ , we have for any  $x \in \mathbb{R}^n$

$$\begin{aligned}
& C(n, s)^{-1} |(-\Delta)^s v_m(x)| \\
& \leq P.V \int_{\mathbb{R}^n} \frac{|v_m(x) - v_m(y)|}{|x - y|^{n+2s}} dy \\
& = P.V. \int_{B_{\varepsilon_0}(x)} \frac{|v_m(x) - v_m(y)|}{|x - y|^{n+2s}} dy + \int_{B_R(x) \setminus B_{\varepsilon_0}(x)} \frac{|v_m(x) - v_m(y)|}{|x - y|^{n+2s}} dy \\
& \quad + \int_{\mathbb{R}^n \setminus B_R(x)} \frac{|v_m(x) - v_m(y)|}{|x - y|^{n+2s}} dy \\
& =: T_1 + T_2 + T_3.
\end{aligned}$$

For the term  $T_1$ , we have

$$\begin{aligned}
T_1 &= \frac{1}{2} P.V. \int_{B_{\varepsilon_0}(0)} \frac{|v_m(x+y) + v_m(x-y) - 2v_m(x)|}{|y|^{n+2s}} dy \\
&\leq C |v_m|_{C^2 B_1(x)} \int_{B_{\varepsilon_0}(0)} |y|^{2-2s-n} dy \leq C |v_m|_{C^2 B_1(x)} \varepsilon_0^{2-2s} \rightarrow 0 \text{ as } \varepsilon_0 \rightarrow 0.
\end{aligned}$$

For the third term,

$$T_3 \leq C \int_{\mathbb{R}^n \setminus B_R(x)} \frac{1}{|x - y|^{n+2s}} = CR^{-2s} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Then we estimate the term  $T_2$ . For fixed  $R$  large enough and  $\varepsilon_0$  small enough,  $B_R(x) \setminus B_{\varepsilon_0}(x)$  is a compact set. So we have  $T_2 \rightarrow 0$  as  $m \rightarrow \infty$ . Hence  $(-\Delta)^s u_m(x) \rightarrow (-\Delta)^s u_\infty(x)$  as  $m \rightarrow \infty$ . Therefore  $u_\infty$  satisfies equation (1.5).  $\square$

*Proof of Corollary 1.3.* Fix the constant  $m < \infty$ . Using a similar argument as in [29], we can expand  $I(W_m + \phi_m)$  as

$$I(W_m + \phi_m) = (m+1)^k \left( \frac{s}{n} \int U_{0,1}^{\frac{2n}{n-2s}} + o(1) \right), \text{ as } l \rightarrow \infty.$$

For each  $m < \infty$ ,  $I(W_m + \phi_m) \rightarrow (m+1)^k \frac{s}{n} \int U_{0,1}^{\frac{2n}{n-2s}}$  as  $l \rightarrow \infty$ . For any  $m_1, m_2 \in \mathbb{N}_+$  such that  $m_1 \neq m_2$ , we can find two solutions  $W_{m_1} + \phi_{m_1}$  and  $W_{m_2} + \phi_{m_2}$  of (1.5), such that  $I(W_{m_1} + \phi_{m_1}) \neq I(W_{m_2} + \phi_{m_2})$ . Hence we can find infinitely many solutions of (1.5).  $\square$

## APPENDIX A. BASIC ESTIMATES

**Lemma A.1.** (cf. [21, 29]) For any  $x_i, x_j, y \in \mathbb{R}^n$  and constant  $\sigma \in [0, \min\{\alpha, \beta\}]$ , we have

$$\frac{1}{(1 + |y - x_i|)^\alpha (1 + |y - x_j|)^\beta} \leq \frac{2^\sigma}{(1 + |x_i - x_j|)^\sigma} \left( \frac{1}{(1 + |y - x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1 + |y - x_j|)^{\alpha+\beta-\sigma}} \right).$$

**Lemma A.2.** For any  $\sigma > 0$  with  $\sigma \neq n - 2s$ , there is a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} \frac{1}{(1 + |z|)^{2s+\sigma}} dz \leq \frac{C}{(1 + |y|)^{\min(\sigma, n-2s)}}.$$

For  $\sigma = n - 2s$ , there is also a constant  $C > 0$ , such that

$$\int_{\mathbb{R}^n} \frac{1}{|y - z|^{n-2s}} \frac{1}{(1 + |z|)^n} \leq C \frac{\max(1, \log |y|)}{(1 + |y|)^{n-2s}}$$

*Proof.* The proof follows from the same argument as [21, Lemma A.2]. See also [29, Lemma B.2].  $\square$

Recall that  $X^i \in X_{i,m} = \{X^i\}_{i=1}^{(m+1)^k}$ ,  $B_i = B_{\lambda l}(X^i)$  and  $B_{i,m} = B_{\max\{\frac{m}{4}, 1\}\lambda l}(X^i)$ .

**Lemma A.3.** (cf. [21]) For any  $\theta > k$ , there exists a constant  $C(\theta, k, n) > 1$  independent of  $m$ , such that if  $y \in B_i \cap \Omega_i$ , there holds

$$\frac{1}{(1 + |y - X^i|)^\theta} \leq \sum_j \frac{1}{(1 + |y - X^j|)^\theta} \leq \frac{C}{(1 + |y - X^i|)^\theta}.$$

If  $y \in B_i^c \cap B_{i,m} \cap \Omega_i$ , there holds

$$\frac{1}{C(1 + |y - X^i|)^{\theta-k}(\lambda l)^k} \leq \sum_j \frac{1}{(1 + |y - X^j|)^\theta} \leq \frac{C}{(1 + |y - X^i|)^{\theta-k}(\lambda l)^k}. \quad (\text{A.1})$$

and if  $y \in B_{i,m}^c \cap \Omega_i$ , there holds

$$\frac{m^k}{C(1 + |y - X^i|)^\theta} \leq \sum_j \frac{1}{(1 + |y - X^j|)^\theta} \leq \frac{Cm^k}{(1 + |y - X^i|)^\theta} \leq \frac{C}{(1 + |y - X^i|)^{\theta-k}(\lambda l)^k}.$$

**Lemma A.4.** Let  $n > 2s + 2$  and  $0 < \tau < \frac{n+2s}{2}$ . If  $\phi$  satisfies  $\|\phi\|_* \leq \frac{C}{\lambda^{\frac{n+2s}{2}-\tau}}$ , then for any  $c > 0$ , there exists  $\lambda_0$  such that for any  $\lambda > \lambda_0$ , there holds  $|\phi| \leq cW_m$  in  $\cup_h(\Omega_h \cap B_h)$ .

*Proof.* We prove this lemma indirectly. Suppose that there exists  $c_0 > 0$ , such that for any  $\lambda_0 > 0$ , there is a  $\lambda > \lambda_0$  and  $y \in \cup_l(\Omega_l \cap B_l)$  such that  $|\phi(y)| \geq c_0 W_m(y)$ . Then

$$\begin{aligned} |\phi(y)| &\geq C \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |y - X^h|)^{n-2s}} \\ &\geq C\gamma(y) \sum_{h=1}^{(m+1)^k} \frac{1}{(1 + |y - X^h|)^{\frac{n-2s}{2}+\tau}} \frac{1}{(\lambda l)^{\frac{n-2s}{2}-\tau}} \end{aligned}$$

If  $0 < \tau < \frac{n-2s}{2}$ , we have

$$\frac{1}{\lambda^{\frac{n+2s}{2}-\tau}} \geq \|\phi\|_* \geq \frac{C}{(\lambda l)^{\frac{n-2s}{2}-\tau}},$$

which does not hold for  $\lambda$  large enough.

If  $\frac{n-2s}{2} \leq \tau < \frac{n+2s}{2}$ , we can also get

$$\frac{1}{\lambda^{\frac{n+2s}{2}-\tau}} \geq \|\phi\|_* \geq C,$$

which also is a contradiction for  $\lambda$  large.  $\square$

**Lemma A.5.** For  $n > 2s + 2$  and  $1 \leq k < \tau < \frac{n-2s}{2}$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2s}} W_{P,\Lambda}^{\frac{4s}{n-2s}}(y) \gamma(y) \sum_h \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau}} dy \\ & \leq C \left( \gamma(x) \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau+\theta}} + \frac{1}{(\lambda l)^{\frac{4s}{n-2s}k}} \gamma(x) \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}} \right), \end{aligned}$$

where  $\theta > 0$  is a small constant and  $C > 0$  does not depend on  $m$ .

*Proof.* Without loss of generality, we assume  $x \in \Omega_1$ . We write

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2s}} W_m^{\frac{4s}{n-2s}}(y) \gamma(y) \sum_h \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau}} dy \\ & = \left( \int_{\cup_h (\Omega_h \cap B_h)} + \int_{\cup_h (\Omega_h \cap B_h^c \cap B_{h,m})} + \int_{\cup_h (\Omega_h \cap B_{h,m}^c)} \right) \frac{1}{|x-y|^{n-2s}} W_m^{\frac{4s}{n-2s}}(y) \\ & \quad \times \gamma(y) \sum_h \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau}} dy \\ & =: T_1 + T_2 + T_3. \end{aligned}$$

We now estimate each term  $T_i$  ( $i = 1, 2, 3$ ).

Using Lemma A.2 and Lemma A.3, we have

$$\begin{aligned} T_1 & \leq C \int_{\cup_h (\Omega_h \cap B_h)} \frac{1}{|x-y|^{n-2s}} \sum_h \frac{1}{(1+|y-X^h|)^{4s+\frac{n-2s}{2}+\tau}} dy \\ & \leq C \sum_h \frac{1}{(1+|x-X^h|)^{\min\{\frac{n+2s}{2}+\tau, n-2s\}}} \\ & = C \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau+\theta_1}}, \quad \text{where } \theta_1 = \min\{2s, \frac{n-2s}{2} - \tau\}. \end{aligned} \quad (\text{A.2})$$

Similarly, we also obtain

$$\begin{aligned} T_1 & \leq C \int_{\cup_h (\Omega_h \cap B_h)} \frac{1}{|x-y|^{n-2s}} \frac{1}{\lambda^{\tau-s}} \sum_h \frac{1}{(1+|y-X^h|)^{\frac{n}{2}+4s}} dy \\ & \leq \frac{C}{\lambda^{\tau-s}} \sum_h \frac{1}{(1+|x-X^h|)^{\min\{\frac{n}{2}+2s, n-2s\}}} \\ & \leq C \frac{(1+|x-X^1|)^{\tau-s}}{\lambda^{\tau-s}} \sum_h \frac{1}{(1+|x-X^h|)^{\min\{n-3s+\tau, \frac{n}{2}+\tau+s\}}} \\ & = C \frac{(1+|x-X^1|)^{\tau-s}}{\lambda^{\tau-s}} \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau+\theta_2}}, \end{aligned} \quad (\text{A.3})$$

where  $\theta_2 = \min\{\frac{n-4s}{2}, 2s\}$ .

Combining the estimate (A.2) and (A.3), we have

$$T_1 \leq C \gamma(x) \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau+\theta}},$$

where  $\theta = \min\{2s, \frac{n-4s}{2}, \frac{n-2s}{2} - \tau\}$ .

For the term  $T_2$ , we have

$$\begin{aligned} T_2 &\leq \frac{C}{(\lambda l)^{\frac{4s}{n-2s}k}} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2s}} \sum_h \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+4s+\tau-\frac{4s}{n-2s}k}} dy \\ &\leq \frac{C}{(\lambda l)^{\frac{4s}{n-2s}k}} \sum_h \frac{1}{(1+|x-X^h|)^{\min\{n-2s, \frac{n-2s}{2}+\tau+2s-\frac{4s}{n-2s}k\}}} \\ &\leq \frac{C}{(\lambda l)^{\frac{4s}{n-2s}k}} \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}}, \end{aligned}$$

and

$$\begin{aligned} T_2 &\leq \frac{C}{\lambda^{\tau-s}(\lambda l)^{\frac{4s}{n-2s}k}} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2s}} \sum_h \frac{1}{(1+|y-X^h|)^{\frac{n}{2}+4s-\frac{4s}{n-2s}k}} dy \\ &\leq \frac{C}{\lambda^{\tau-s}(\lambda l)^{\frac{4s}{n-2s}k}} \sum_h \frac{1}{(1+|x-X^h|)^{\min\{n-2s, \frac{n}{2}+2s-\frac{4s}{n-2s}k\}}} \\ &\leq \frac{C}{(\lambda l)^{\frac{4s}{n-2s}k}} \frac{(1+|x-X^1|)^{\tau-s}}{\lambda^{\tau-s}} \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}}. \end{aligned}$$

Thus

$$T_2 \leq \frac{C}{(\lambda l)^{\frac{4s}{n-2s}k}} \gamma(x) \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}}.$$

By the same procedure, we have

$$T_3 \leq \frac{C}{(\lambda l)^{\frac{4s}{n-2s}k}} \gamma(x) \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}}.$$

Hence this lemma follows.  $\square$

Remember  $Z_{i,j} = \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}$  for  $j = 1, 2, \dots, n$  and  $Z_{i,n+1} = \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i}$ .

**Lemma A.6.** For  $t = 1, 2, \dots, n+1$ , we have

$$\left| \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \leq \frac{C \|\phi\|_*}{\lambda^{\tau-s} (\lambda l)^{\frac{n}{2}}}.$$

*Proof.* In the proof of this lemma, we denote  $\hat{W}_{m,r} = \sum_{h \neq r} U_{P^h, \Lambda_h}$ . It is easy to get

$$\begin{aligned} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}} Z_{r,t} \phi &= \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^r, \Lambda_r}^{\frac{4s}{n-2s}} Z_{r,t} \phi + O\left(\int_{\hat{W}_{m,r} > U_{P^r, \Lambda_r}} \hat{W}_{m,r}^{\frac{4s}{n-2s}} Z_{r,t} \phi\right) \\ &\quad + O\left(\int_{\hat{W}_{m,r} \leq U_{P^r, \Lambda_r}} U_{P^r, \Lambda_r}^{\frac{4s}{n-2s}} \hat{W}_{m,r} \phi\right). \end{aligned} \quad (\text{A.4})$$

We need to estimate each term in the equality above.

For  $i \neq r$ , from Lemma A.3, we have

$$\begin{aligned}
& \left| \int_{\Omega_i \cap B_i} \hat{W}_{m,r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \\
& \leq C \frac{\|\phi\|_*}{\lambda^{\tau-s}} \int_{\Omega_i \cap B_i} \left( \sum_{h \neq r} \frac{1}{(1+|x-X^h|)^{n-2s}} \right)^{\frac{4s}{n-2s}} \frac{1}{(1+|x-X^r|)^{n-2s}} \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n}{2}}} dx \\
& \leq C \frac{\|\phi\|_*}{\lambda^{\tau-s}} \int_{\Omega_i \cap B_i} \frac{1}{(1+|x-X^r|)^{n-2s}} \frac{1}{(1+|x-X^i|)^{\frac{n}{2}+4s}} dx \leq C \frac{\|\phi\|_*}{\lambda^{\tau-s} |X^r - X^i|^{\frac{n}{2}}}.
\end{aligned}$$

With the help of Lemma A.1 and Lemma A.3, we get

$$\begin{aligned}
& \left| \int_{\cup_h (\Omega_h \cap B_h^c)} \hat{W}_{m,r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \\
& \leq C \|\phi\|_* \int_{\cup_h (\Omega_h \cap B_h^c)} \frac{1}{(1+|x-X^r|)^{n-2s}} \left( \sum_{h \neq r} \frac{1}{(1+|x-X^h|)^{n-2s}} \right)^{\frac{4s}{n-2s}} \\
& \quad \times \left( \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n-2s}{2}+\tau}} \right) dx \tag{A.5} \\
& \leq C \frac{\|\phi\|_*}{(\lambda l)^{\frac{4s}{n-2s}k}} \int_{\cup_h (\Omega_h \cap B_h^c)} \frac{1}{(1+|x-X^r|)^{n-2s}} \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n}{2}+3s+\tau-\frac{4s}{n-2s}k}} dx \\
& \leq C \frac{\|\phi\|_*}{(\lambda l)^{\frac{n}{2}+s+\tau}}.
\end{aligned}$$

Since in  $\Omega_r \cap B_r$ , there holds  $\hat{W}_{m,r} \leq \sum_{j \neq r} \frac{C}{|X^j - X^r|^{n-2s}} \leq \frac{C}{(\lambda l)^{n-2s}}$ . Then if  $n \geq 6s$ , we have

$$\begin{aligned}
& \left| \int_{\substack{\Omega_r \cap B_r \\ \hat{W}_{m,r} > U_{P^r, \Lambda_r}}} \hat{W}_{m,r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \leq C \int_{\Omega_r \cap B_r} \hat{W}_{m,r}^{\frac{n+2s}{2(n-2s)}} U_{P^r, \Lambda_r}^{\frac{n+2s}{2(n-2s)}} |\phi| \\
& \leq C \frac{\|\phi\|_*}{\lambda^{\tau-s}} \frac{1}{(\lambda l)^{\frac{n+2s}{2}}} \int_{\Omega_r \cap B_r} \frac{1}{(1+|x-X^r|)^{\frac{n+2s}{2}}} \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n}{2}}} dx \\
& \leq C \frac{\|\phi\|_*}{\lambda^{\tau-s} (\lambda l)^{\frac{n+2s}{2}}}.
\end{aligned}$$

And if  $n < 6s$ , we get  $\frac{n+2s}{2} < 4s$ . In this case

$$\begin{aligned}
& \left| \int_{\substack{\Omega_r \cap B_r \\ \hat{W}_{m,r} > U_{P^r, \Lambda_r}}} \hat{W}_{m,r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \leq \frac{C \|\phi\|_*}{\lambda^{\tau-s} (\lambda l)^{4s}} \int_{\Omega_r \cap B_r} \frac{1}{(1+|x-X^r|)^{n-2s}} \sum_h \frac{1}{(1+|x-X^h|)^{\frac{n}{2}}} dx \\
& \leq \frac{C \|\phi\|_*}{\lambda^{\tau-s} (\lambda l)^{\frac{n+2s}{2}}}.
\end{aligned}$$

From these arguments above, we arrive

$$\left| \int_{\hat{W}_{m,r} > U_{Pr, \Lambda_r}} \hat{W}_{m,r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \leq C \frac{\|\phi\|_*}{\lambda^{\tau-s} (\lambda l)^{\frac{n}{2}}}. \quad (\text{A.6})$$

By a similar procedure, we get

$$\left| \int_{\hat{W}_{m,r} \leq U_{Pr, \Lambda_r}} U_{Pr, \Lambda_r}^{\frac{4s}{n-2s}} \hat{W}_{m,r} \phi \right| \leq C \frac{\|\phi\|_*}{\lambda^{\tau-s} (\lambda l)^{\frac{n}{2}}}. \quad (\text{A.7})$$

Now we estimate the first term on the right hand side of the equality (A.4). Since  $\phi$  satisfies the second equality in (2.10), we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{Pr, \Lambda_r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| = \left| \int_{\mathbb{R}^n} \left( K\left(\frac{x}{\lambda}\right) - 1 \right) U_{Pr, \Lambda_r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \\ & \leq \frac{\|\phi\|_*}{\lambda^{\tau-s}} \int_{\mathbb{R}^n} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{Pr, \Lambda_r}^{\frac{n+2s}{n-2s}} \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n}{2}}} dx. \end{aligned}$$

On one hand, Lemma A.1 implies that

$$\begin{aligned} & \int_{\mathbb{R}^n} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{Pr, \Lambda_r}^{\frac{n+2s}{n-2s}} \sum_{h \neq r} \frac{1}{(1 + |x - X^h|)^{\frac{n}{2}}} dx \\ & \leq C \int_{\mathbb{R}^n} \frac{1}{(1 + |x - X^r|)^{n+2s}} \sum_{h \neq r} \frac{1}{(1 + |x - X^h|)^{\frac{n}{2}}} dx \\ & \leq \frac{C}{(\lambda l)^{\frac{n}{2}}}. \end{aligned}$$

On the another hand, choose  $\delta$  to be a fixed constant small enough,

$$\begin{aligned} & \int_{\mathbb{R}^n} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{Pr, \Lambda_r}^{\frac{n+2s}{n-2s}} \frac{1}{(1 + |x - X^r|)^{\frac{n}{2}}} \\ & = \int_{|x - X^r| \leq \delta \lambda} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{Pr, \Lambda_r}^{\frac{n+2s}{n-2s}} \frac{1}{(1 + |x - X^r|)^{\frac{n}{2}}} \\ & \quad + \int_{|x - X^r| > \delta \lambda} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{Pr, \Lambda_r}^{\frac{n+2s}{n-2s}} \frac{1}{(1 + |x - X^r|)^{\frac{n}{2}}} =: J_1 + J_2. \end{aligned}$$

From the condition  $(H_3)$ , we have

$$|J_1| \leq \frac{C}{\lambda^\beta} \int_{|x - X^r| \leq \delta \lambda} \frac{|x - X^r|^\beta}{(1 + |x - X^r|)^{n+2s+\frac{n}{2}}} \leq \begin{cases} \frac{C \log \lambda}{\lambda^{\frac{n+4s}{2}}}, & \text{if } \beta \geq \frac{n+4s}{2}, \\ \frac{C}{\lambda^\beta}, & \text{if } \beta < \frac{n+4s}{2}. \end{cases}$$

For the term  $J_2$ , a direct calculation yields

$$\begin{aligned} J_2 &= \int_{|x-X^r|>\delta\lambda} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{P^r, \Lambda_r}^{\frac{n+2s}{n-2s}} \frac{1}{(1+|x-X^r|)^{\frac{n}{2}}} \\ &\leq \int_{|x-X^r|>\delta\lambda} \frac{1}{(1+|x-X^r|)^{n+2s+\frac{n}{2}}} \\ &\leq \frac{C}{\lambda^{\frac{n+4s}{2}}}. \end{aligned}$$

Since  $\min\{\beta, \frac{n+4s}{2}\} > \frac{n}{2} \frac{\beta}{n-2s}$ , the definition of  $\lambda$  implies

$$\int_{\mathbb{R}^n} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{P^r, \Lambda_r}^{\frac{n+2s}{n-2s}} \frac{1}{(1+|x-X^r|)^{\frac{n}{2}}} \leq \frac{C}{(\lambda l)^{\frac{n}{2}}}.$$

Hence we obtain

$$\left| \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^r, \Lambda_r}^{\frac{4s}{n-2s}} Z_{r,t} \phi \right| \leq C \frac{\|\phi\|_*}{\lambda^{\tau-s} (\lambda l)^{\frac{n}{2}}}. \quad (\text{A.8})$$

Putting (A.6), (A.7) and (A.8) into (A.4), we get this lemma.  $\square$

**Lemma A.7.** *It holds that*

$$\int_{\cup_h (\Omega_h \cap B_h^c)} |\phi|^{\frac{n+2s}{n-2s}} U_{P^i, \Lambda_i} \leq C \frac{\|\phi\|_*^{\frac{n+2s}{n-2s}}}{(\lambda l)^{\frac{n-2s}{2} + \frac{n+2s}{n-2s} \tau}}, \quad (\text{A.9})$$

and

$$\left| \int_{\mathbb{R}^n} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i, \Lambda_i} \right| \leq \frac{C \|\phi\|_*^2}{\lambda^{2(\tau-s)}}. \quad (\text{A.10})$$

*Proof.* The estimate (A.9) follows by the same method as in (A.5). So we only prove the estimation (A.10).

Using the same trick as in (A.5), we have

$$\left| \int_{\cup_h (\Omega_h \cap B_h^c)} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i, \Lambda_i} \right| \leq \frac{C \|\phi\|_*^2}{\lambda^{2(\tau+s)}}. \quad (\text{A.11})$$

According to Lemma A.3, we have for  $t \neq i$ ,

$$\left| \int_{\Omega_t \cap B_t} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i, \Lambda_i} \right| \leq \frac{C \|\phi\|_*^2}{(\lambda l)^{2(\tau-s)}} \int_{\Omega_t \cap B_t} \frac{1}{(1+|y-X^t|)^{6s}} \frac{1}{(1+|y-X^i|)^{n-2s}} dy.$$

If  $n \geq 6s$

$$\int_{\Omega_t \cap B_t} \frac{1}{(1+|y-X^t|)^{6s}} \frac{1}{(1+|y-X^i|)^{n-2s}} dy \leq \frac{(\lambda l)^{n-6s} \log(\lambda l)}{|X^i - X^t|^{n-2s}};$$

If otherwise,  $n < 6s$ , there holds

$$\int_{\Omega_t \cap B_t} \frac{1}{(1+|y-X^t|)^{6s}} \frac{1}{(1+|y-X^i|)^{n-2s}} dy \leq \frac{1}{|X^i - X^t|^{n-2s}}.$$

Hence

$$\sum_{t \neq i} \left| \int_{\Omega_t \cap B_t} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i, \Lambda_i} \right| \leq \frac{C \|\phi\|_*^2}{(\lambda l)^{2(\tau-s)}} \frac{C \log(\lambda l)}{(\lambda l)^{\min\{n-2s, 4s\}}}. \quad (\text{A.12})$$



Using Lemma A.3, we also have

$$\left| \int_{\Omega_i \cap B_i} W_m^{\frac{6s-n}{n-2s}} \phi^2 U_{P^i, \Lambda_i} \right| \leq \frac{C \|\phi\|_*^2}{(\lambda l)^{2(\tau-s)}} \int_{\Omega_i \cap B_i} \frac{1}{(1 + |y - X^i|)^{n+4s}}. \quad (\text{A.13})$$

So we obtain the estimate (A.10) from (A.11), (A.12) and (A.13).  $\square$

**Lemma A.8.** *If  $\phi$  is the solution of the equation*

$$(-\Delta)^s \phi(x) - \frac{n+2s}{n-2s} K\left(\frac{x}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(x) \phi(x) = g(x), \quad (\text{A.14})$$

satisfying  $\|\phi\|_* < +\infty$ , then we have

$$\sup_{x_1 \neq x_2} \frac{|\lambda^{\tau-s} \phi(x_1) - \lambda^{\tau-s} \phi(x_2)|}{|x_1 - x_2|^\alpha} \leq C \max\{\|\phi\|_*, \|g\|_{**}\},$$

where  $\alpha = \min\{2s, 1\}$  and the constant  $C$  does not depend on  $\lambda$  and  $m$ .

*Proof.* Since  $|\lambda^{\tau-s} \phi(x)| \leq \|\phi\|_* \sum_h \frac{1}{(1 + |x - X^h|)^{\frac{n}{2}}} \leq C \|\phi\|_*$ , we can assume  $|x_1 - x_2| \leq \frac{1}{3}$  with no loss of generality. Using the Green function of  $(-\Delta)^s$  (see [5]), we can write (A.14) into the following form

$$\phi(x) = C \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2s}} \left( \frac{n+2s}{n-2s} K\left(\frac{y}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(y) \phi(y) + g(y) \right) dy,$$

Then we get

$$\begin{aligned} |\phi(x_1) - \phi(x_2)| &\leq C \left| \int_{\mathbb{R}^n} \left( \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right) K\left(\frac{y}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(y) \phi(y) dy \right| \\ &\quad + C \left| \int_{\mathbb{R}^n} \left( \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right) g(y) dy \right| \\ &=: C(H_1 + H_2). \end{aligned}$$

Using the definition of the norm  $\|\cdot\|_*$ , there hold

$$\begin{aligned}
|H_1| &\leq \|\phi\|_* \left| \int_{\mathbb{R}^n} \left( \frac{1}{|x_1 - x_2 - y|^{n-2s}} - \frac{1}{|y|^{n-2s}} \right) K\left(\frac{y+x_2}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(y+x_2) \times \right. \\
&\quad \left. \times \gamma(y+x_2) \sum_h \frac{1}{(1+|y+x_2-X^h|)^{\frac{n-2s}{2}+\tau}} dy \right| \\
&= \|\phi\|_* \left\{ \int_{|y| \leq 3|x_1-x_2|} \left( \frac{1}{|x_1 - x_2 - y|^{n-2s}} - \frac{1}{|y|^{n-2s}} \right) K\left(\frac{y+x_2}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(y+x_2) \times \right. \\
&\quad \times \gamma(y+x_2) \sum_h \frac{1}{(1+|y+x_2-X^h|)^{\frac{n-2s}{2}+\tau}} dy \\
&\quad + \int_{|y| \geq 3|x_1-x_2|} \left( \frac{1}{|x_1 - x_2 - y|^{n-2s}} - \frac{1}{|y|^{n-2s}} \right) K\left(\frac{y+x_2}{\lambda}\right) W_m^{\frac{4s}{n-2s}}(y+x_2) \times \\
&\quad \left. \times \gamma(y+x_2) \sum_h \frac{1}{(1+|y+x_2-X^h|)^{\frac{n-2s}{2}+\tau}} dy \right\} \\
&=: \|\phi\|_*(K_1 + K_2).
\end{aligned}$$

For the term  $K_1$ , we have

$$|K_1| \leq \frac{C}{\lambda^{\tau-s}} \int_{|y| \leq 4|x_1-x_2|} \frac{1}{|y|^{n-2s}} \leq \frac{C}{\lambda^{\tau-s}} |x_1 - x_2|^{2s}.$$

For the term  $K_2$ , we have

$$\begin{aligned}
|K_2| &\leq C|x_1 - x_2| \int_0^1 dt \int_{|y| \geq 3|x_1-x_2|} \frac{1}{|t(x_1 - x_2) - y|^{n-2s+1}} W_m^{\frac{4s}{n-2s}}(y+x_2) \\
&\quad \times \gamma(y+x_2) \sum_h \frac{1}{(1+|y+x_2-X^h|)^{\frac{n-2s}{2}+\tau}} dy \\
&= C|x_1 - x_2| \int_0^1 dt \left( \int_{1 > |y| \geq 3|x_1-x_2|} + \int_{|y| \geq 1} \right) \frac{1}{|t(x_1 - x_2) - y|^{n-2s+1}} W_m^{\frac{4s}{n-2s}}(y+x_2) \\
&\quad \times \gamma(y+x_2) \sum_h \frac{1}{(1+|y+x_2-X^h|)^{\frac{n-2s}{2}+\tau}} dy \\
&=: C|x_1 - x_2|(M_1 + M_2).
\end{aligned}$$

Since it holds that  $|t(x_1 - x_2) - y| \in [2|x_1 - x_2|, \frac{4}{3}]$  for  $1 > |y| \geq 3|x_1 - x_2|$ , we get

$$|M_1| \leq \frac{1}{\lambda^{\tau-s}} \int_{2|x_1-x_2| \leq |y| \leq \frac{4}{3}} \frac{1}{|y|^{n-2s+1}} \leq \frac{1}{\lambda^{\tau-s}} (C + C|x_1 - x_2|^{2s-1}).$$

For  $|y| \geq 1$ , we have  $\frac{2}{3}|y| \leq |t(x_1 - x_2) - y| \leq \frac{4}{3}|y|$ . Lemma A.5 yields

$$\begin{aligned} M_2 &\leq C \int_{|y| \geq 1} \frac{1}{|y|^{n-2s+1}} W_m^{\frac{4s}{n-2s}}(y+x_2) \gamma(y+x_2) \sum_h \frac{1}{(1+|y+x_2-X^h|)^{\frac{n-2s}{2}+\tau}} dy \\ &\leq C \int_{\mathbb{R}^n} \frac{1}{|y-x_2|^{n-2s}} W_m^{\frac{4s}{n-2s}}(y) \gamma(y) \sum_h \frac{1}{(1+|y-X^h|)^{\frac{n-2s}{2}+\tau}} dy \\ &\leq \frac{C}{\lambda^{\tau-s}}. \end{aligned}$$

Hence  $|H_1| \leq \frac{C}{\lambda^{\tau-s}} \|\phi\|_* |x_1 - x_2|^\alpha$ . The same procedure with the help of Lemma A.2 yields that  $|H_2| \leq \frac{C}{\lambda^{\tau-s}} \|g\|_{**} |x_1 - x_2|^\alpha$ . Then Lemma A.8 follows.  $\square$

## APPENDIX B. EXPANSIONS OF THE FUNCTIONALS $\frac{\partial}{\partial \Lambda_i} I(W_m)$ AND $\frac{\partial}{\partial P_j^i} I(W_m)$

In this section, we will expand the functionals  $\frac{\partial}{\partial \Lambda_i} I(W_m)$  and  $\frac{\partial}{\partial P_j^i} I(W_m)$ . A direct computation yields

$$\begin{aligned} \frac{\partial I}{\partial \Lambda_i}(W_m) &= \int_{\mathbb{R}^n} W_m(-\Delta)^s \frac{\partial W_m}{\partial \Lambda_i} - \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial W_m}{\partial \Lambda_i} \\ &= \int_{\mathbb{R}^n} \sum_{h=1}^{(m+1)^k} U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} - \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i}, \end{aligned} \quad (\text{B.1})$$

and

$$\begin{aligned} \frac{\partial I}{\partial P_j^i}(W_m) &= \int_{\mathbb{R}^n} W_m(-\Delta)^s \frac{\partial W_m}{\partial P_j^i} - \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial W_m}{\partial P_j^i} \\ &= \int_{\mathbb{R}^n} \sum_{h \neq i} U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} - \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}. \end{aligned} \quad (\text{B.2})$$

In order to get the useful expansions, we need to estimate each term on the right hand side of (B.1) and (B.2) above.

**Lemma B.1.** *There holds*

$$\begin{aligned} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} &= \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) \sum_h U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\ &\quad + \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} \sum_{h \neq i} U_{P^h, \Lambda_h} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + O((\lambda l)^{-n}). \end{aligned}$$

*Proof.* We estimate the integration on different region. By the same method used in (A.5), we have

$$\int_{\cup_h (\Omega_h \cap B_h^c)} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \left| \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \right| = O((\lambda l)^{-n}). \quad (\text{B.3})$$

In the domain  $\Omega_j \cap B_j$ , where  $j \neq i$ , there holds  $\hat{W}_{m,j}(y) \leq \sum_{h \neq j} \frac{C}{|X^j - X^h|} \leq \frac{C}{(\lambda l)^{n-2s}} \leq CU_{P^j, \Lambda_j}$ . Taylor expansion yields

$$\begin{aligned} & \int_{\Omega_j \cap B_j} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\ &= \int_{\Omega_j \cap B_j} K\left(\frac{x}{\lambda}\right) U_{P^j, \Lambda_j}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + O\left(\int_{\Omega_j \cap B_j} U_{P^j, \Lambda_j}^{\frac{4s}{n-2s}} \hat{W}_{m,j} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i}\right). \end{aligned}$$

For the error term, a direct computation yields

$$\int_{\Omega_j \cap B_j} U_{P^j, \Lambda_j}^{\frac{4s}{n-2s}} \hat{W}_{m,j} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = O\left(\frac{1}{(\lambda l)^{2s} |X^i - X^j|^{n-2s}}\right).$$

**Claim:** For  $j \neq i$ , there holds

$$\int_{\Omega_j \cap B_j} K\left(\frac{x}{\lambda}\right) U_{P^j, \Lambda_j}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^j, \Lambda_j}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + O\left(\frac{1}{(\lambda l)^{2s} |X^i - X^j|^{n-2s}}\right). \quad (\text{B.4})$$

From direct computation

$$\int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) U_{P^j, \Lambda_j}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = O\left(\frac{(\lambda l)^{2s}}{|X^i - X^j|^{n+2s}}\right). \quad (\text{B.5})$$

Using Lemma A.1, we can obtain

$$\begin{aligned} \sum_{h \neq i, j} \int_{\Omega_h \cap B_h} K\left(\frac{x}{\lambda}\right) U_{P^j, \Lambda_j}^{\frac{n+2s}{n-2s}} \left| \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \right| &\leq \sum_{h \neq i, j} \frac{C}{|X^i - X^j|^{n-2s}} \int_{\Omega_h \cap B_h} \frac{1}{(1 + |x - X^i|)^{n+2s}} \\ &\leq \frac{C}{|X^i - X^j|^{n-2s}} \sum_{h \neq i} \frac{(\lambda l)^n}{|X^h - X^i|^{n+2s}} \\ &\leq \frac{C}{(\lambda l)^{2s} |X^i - X^j|^{n-2s}}, \end{aligned} \quad (\text{B.6})$$

and

$$\int_{\cup_h (\Omega_h \cap B_h^c)} K\left(\frac{x}{\lambda}\right) U_{P^j, \Lambda_j}^{\frac{n+2s}{n-2s}} \left| \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \right| = O\left(\frac{1}{(\lambda l)^{2s} |X^i - X^j|^{n-2s}}\right). \quad (\text{B.7})$$

From (B.5), (B.6) and (B.7), we know the Claim is true.

Hence for  $j \neq i$ ,

$$\int_{\Omega_j \cap B_j} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^j, \Lambda_j}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + O\left(\frac{1}{(\lambda l)^{2s} |X^i - X^j|^{n-2s}}\right). \quad (\text{B.8})$$

Now we estimate the integration on  $\Omega_i \cap B_i$ . By Taylor expansion,

$$\begin{aligned}
& \int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\
&= \int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + \frac{n+2s}{n-2s} \int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} \sum_{h \neq i} U_{P^h, \Lambda_h} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\
&\quad + O\left(\int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} \hat{W}_{m,i}^2\right) \\
&= \int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + \frac{n+2s}{n-2s} \int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} \sum_{h \neq i} U_{P^h, \Lambda_h} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\
&\quad + O((\lambda l)^{-n}). \tag{B.9}
\end{aligned}$$

Since in the domain  $\Omega_i^c \cup B_i^c$ , we have  $|y - X^i| \geq \min\{\lambda l, \min_{j \neq i} \frac{1}{2}|X^i - X^j|\} \geq \frac{1}{2}\lambda l$ . Then

$$\int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + O((\lambda l)^{-n}). \tag{B.10}$$

By a similar method used in the proof of (B.4), we get

$$\begin{aligned}
\int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} U_{P^j, \Lambda_j} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} &= \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} U_{P^j, \Lambda_j} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\
&\quad + O\left(\frac{1}{(\lambda l)^{2s}|X^i - X^j|^{n-2s}}\right). \tag{B.11}
\end{aligned}$$

Substituting (B.10) and (B.11) into (B.9), we have

$$\begin{aligned}
& \int_{\Omega_i \cap B_i} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\
&= \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} \sum_{h \neq i} U_{P^h, \Lambda_h} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\
&\quad + O((\lambda l)^{-n}). \tag{B.12}
\end{aligned}$$

Now Lemma B.1 follows from the estimate (B.3), (B.8) and (B.12).  $\square$

**Lemma B.2.** *For  $h \neq i$ , there holds*

$$\int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = \int_{\mathbb{R}^n} U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + O\left(\frac{1}{\lambda^{2s}|X^i - X^h|^{n-2s}}\right), \tag{B.13}$$

and

$$\frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} U_{P^h, \Lambda_h} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = \int_{\mathbb{R}^n} U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} + O\left(\frac{1}{\lambda^{2s}|X^i - X^h|^{n-2s}}\right). \tag{B.14}$$

*Proof.* Notice the fact

$$\begin{aligned}
\int_{\mathbb{R}^n} U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} &= \int_{\mathbb{R}^n} (-\Delta)^s U_{P^h, \Lambda_h} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\
&= \int_{\mathbb{R}^n} U_{P^h, \Lambda_h} (-\Delta)^s \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \\
&= \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} U_{P^h, \Lambda_h} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i}.
\end{aligned}$$

So the proof of (B.14) and (B.13) are identical. We only give a proof of (B.13).

Choose  $\delta$  to be a fixed constant some enough. Since  $n > 4s > n + 2s - \beta$ , the condition  $(H_3)$  implies

$$\begin{aligned}
&\left| \int_{B_{\delta\lambda}(X^h)} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \right| \\
&\leq C \int_{B_{\delta\lambda}(X^h)} \frac{|x - X^h|^\beta}{\lambda^\beta} \frac{1}{(1 + |x - P^h|)^{n+2s}} \frac{1}{(1 + |x - P^i|)^{n-2s}} \\
&\leq \frac{C}{\lambda^\beta |X^i - X^h|^{n-2s}} \int_{B_{\delta\lambda}(X^h)} \frac{|x - X^h|^\beta}{(1 + |x - X^h|)^{n+2s}} \leq \frac{C}{\lambda^{2s} |X^i - X^h|^{n-2s}}. \tag{B.15}
\end{aligned}$$

A direct calculation yields

$$\left| \int_{B_{\delta\lambda}(X^i)} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \right| \leq \frac{C\lambda^{2s}}{|X^i - X^h|^{n+2s}} \leq \frac{C}{\lambda^{2s} |X^i - X^h|^{n-2s}}. \tag{B.16}$$

Using Lemma A.1, we have

$$\begin{aligned}
&\left| \int_{B_{\delta\lambda}^c(X^h) \cap B_{\delta\lambda}^c(X^i)} \left| K\left(\frac{x}{\lambda}\right) - 1 \right| U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} \right| \\
&\leq C \int_{B_{\delta\lambda}^c(X^h) \cap B_{\delta\lambda}^c(X^i)} \frac{1}{(1 + |x - X^h|)^{n+2s}} \frac{1}{(1 + |x - X^i|)^{n-2s}} \\
&\leq \frac{C}{|X^h - X^i|^{n-2s}} \int_{B_{\delta\lambda}^c(X^h)} \frac{1}{(1 + |x - X^h|)^{n+2s}} \leq \frac{C}{\lambda^{2s} |X^i - X^h|^{n-2s}}. \tag{B.17}
\end{aligned}$$

Hence (B.13) follows from (B.15), (B.16) and (B.17). □

**Lemma B.3.** *We have*

$$\begin{aligned}
\int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} &= -\frac{n-2s}{2n} \frac{\beta C_0(n, s)^{\frac{2n}{n-2s}} (\sum_h a_h)}{\Lambda_i^{\beta+1} \lambda^\beta} \int_{\mathbb{R}^n} \frac{|x_1|^\beta}{(1 + |x|^2)^n} \\
&\quad + O\left(\frac{|P^i - X^i|^{\min\{2, \beta-1\}}}{\lambda^\beta}\right) + o\left(\lambda^{-\beta}\right), \tag{B.18}
\end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} &= \frac{(n-2s)C_0(n, s)^{\frac{2n}{n-2s}} \beta a_j}{\Lambda_i^{\beta-2} \lambda^\beta} \int_{\mathbb{R}^n} \frac{|x_1|^\beta}{(1+|x|^2)^{n+1}} (P_j^i - X_j^i) \\ &\quad + O\left(\frac{|P^i - X^i|^2}{\lambda^\beta}\right) + o(\lambda^{-\beta}), \end{aligned} \quad (\text{B.19})$$

where  $i = 1, \dots, (m+1)^k$  and  $j = 1, \dots, n$ .

*Proof.* The two formulas follows from some standard calculations, see [21, Lemma A.9, Lemma A.10] for details.  $\square$

**Lemma B.4.** For  $h \neq i$ , we have

$$\int_{\mathbb{R}^n} U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial \Lambda_i} = c_0 \frac{\partial \varepsilon_{ih}}{\partial \Lambda_i} + \frac{1}{\Lambda_i} O(\varepsilon_{hi}^{\frac{n}{n-2s}} \log \varepsilon_{hi}),$$

where  $c_0 = C_0(n, s)^{\frac{2n}{n-2s}} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2s}{2}}}$  and  $\varepsilon_{ih} = \left( \frac{1}{\frac{\Lambda_i}{\Lambda_h} + \frac{\Lambda_h}{\Lambda_i} + \Lambda_i \Lambda_h |P^i - P^h|^2} \right)^{\frac{n-2s}{2}}$ .

*Proof.* The proof of this lemma is rather standard. We refer to [3] and [9] for ideas.  $\square$

**Proposition B.5.** It holds that

$$\frac{\partial I}{\partial \Lambda_i}(W_m) = -\frac{c_1}{\Lambda_i^{\beta+1} \lambda^\beta} + \sum_{h \neq i} \frac{c_2}{\Lambda_i(\Lambda_i \Lambda_h)^{\frac{n-2s}{2}} |X^i - X^h|^{n-2s}} + O\left(\frac{|P^i - X^i|^{\min\{2, \beta-1\}}}{\lambda^\beta}\right) + o(\lambda^{-\beta}),$$

where  $c_1 = \frac{(n-2s)\beta C_0(n, s)^{\frac{2n}{n-2s}} (-\sum_h a_h)}{2n} \int_{\mathbb{R}^n} \frac{|x_1|^\beta}{(1+|x|^2)^n} > 0$  and  $c_2 = \frac{n-2s}{2} C_0(n, s)^{\frac{2n}{n-2s}} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2s}{2}}}$ .

*Proof.* This proposition is a consequence of Lemma B.1, (B.13), (B.14), (B.18), Lemma B.4 and the definition of  $\lambda$ . We need to remind that

$$\frac{\partial \varepsilon_{ih}}{\partial \Lambda_i} = -\frac{n-2s}{2\Lambda_i(\Lambda_i \Lambda_h)^{\frac{n-2s}{2}} |X^i - X^h|^{n-2s}} + O\left(\frac{1}{|X^i - X^h|^{n-2s+1}}\right),$$

which is directly from  $P_h \in B_{\frac{1}{2}}(X^h)$  and the definition of  $\{X^h\}_{h=1}^{(m+1)^k}$ .  $\square$

**Proposition B.6.** We have

$$\frac{\partial I}{\partial P_j^i}(W_m) = -\frac{c_3 a_j}{\Lambda_i^{\beta-2} \lambda^\beta} (P_j^i - X_j^i) + O\left(\frac{|P^i - X^i|^2}{\lambda^\beta}\right) + o(\lambda^{-\beta}),$$

where  $c_3 = (n-2s)C_0(n, s)^{\frac{2n}{n-2s}} \beta \int \frac{|x_1|^\beta}{(1+|x|^2)^{n+1}}$ .

*Proof.* We need to estimate each term on the right hand side of the equality (B.2). By simple calculation, we have

$$\left| W_m^{\frac{n+2s}{n-2s}} - U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} - \frac{n+2s}{n-2s} U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} \sum_{h \neq i} U_{P^h, \Lambda_h} \right| \leq \begin{cases} \left( \sum_{h \neq i} U_{P^h, \Lambda_h} \right)^{\frac{n+2s}{n-2s}}, & \text{if } U_{P^i, \Lambda_i} \leq \sum_{h \neq i} U_{P^h, \Lambda_h}, \\ U_{P^i, \Lambda_i}^{\frac{6s-n}{n-2s}} \left( \sum_{h \neq i} U_{P^h, \Lambda_h} \right)^2, & \text{otherwise.} \end{cases}$$

Then we get

$$\begin{aligned}
& \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) W_m^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \\
&= \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} + \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} \sum_{h \neq i} U_{P^h, \Lambda_h} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \\
&+ O\left(\int_{\mathbb{R}^n} \left(\sum_{h \neq i} U_{P^h, \Lambda_h}\right)^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}\right) + O\left(\int_{U_{P^i, \Lambda_i} > \sum_{h \neq i} U_{P^h, \Lambda_h}} U_{P^i, \Lambda_i}^{\frac{6s-n}{n-2s}} \left(\sum_{h \neq i} U_{P^h, \Lambda_h}\right)^2 \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}\right).
\end{aligned} \tag{B.20}$$

We first estimate the error terms above. Since  $n > 2s + 2 > 4s$ , we get  $(n-s)\frac{n-2s}{n+2s} > \frac{n-2s}{2} > k$ . From Lemma A.1 we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(\sum_{h \neq i} U_{P^h, \Lambda_h}\right)^{\frac{n+2s}{n-2s}} \left|\frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}\right| &\leq C \int_{\mathbb{R}^n} \left(\sum_{h \neq i} \frac{1}{(1+|y-X^h|)^{n-2s}}\right)^{\frac{n+2s}{n-2s}} \frac{1}{(1+|y-X^i|)^{n-2s+1}} \\
&\leq C \int_{\mathbb{R}^n} \left(\sum_{h \neq i} \frac{1}{(1+|y-X^h|)^{n-2s}} \frac{1}{(1+|y-X^i|)^{(n-s)\frac{n-2s}{n+2s}}}\right)^{\frac{n+2s}{n-2s}} \\
&\leq \left(\sum_{h \neq i} \frac{1}{|X^h - X^i|^{(n-s)\frac{n-2s}{n+2s}}}\right)^{\frac{n+2s}{n-2s}} \int_{\mathbb{R}^n} \frac{1}{(1+|y-X^i|)^{n+2s}} \\
&\leq C(\lambda)^{-(n-s)}.
\end{aligned} \tag{B.21}$$

The similar argument yields

$$\begin{aligned}
\int_{U_{P^i, \Lambda_i} > \sum_{h \neq i} U_{P^h, \Lambda_h}} U_{P^i, \Lambda_i}^{\frac{6s-n}{n-2s}} \left(\sum_{h \neq i} U_{P^h, \Lambda_h}\right)^2 \left|\frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}\right| &\leq \int_{\mathbb{R}^n} \left(\sum_{h \neq i} U_{P^h, \Lambda_h}\right)^{\frac{n}{n-2s}} U_{P^i, \Lambda_i}^{\frac{2s}{n-2s}} \left|\frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i}\right| \\
&\leq C(\lambda)^{-n}.
\end{aligned} \tag{B.22}$$

For  $h \neq i$ , we see that

$$\begin{aligned}
& \frac{n+2s}{n-2s} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{4s}{n-2s}} U_{P^h, \Lambda_h} \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \\
&= \frac{\partial}{\partial P_j^i} \int_{\mathbb{R}^n} K\left(\frac{x}{\lambda}\right) U_{P^i, \Lambda_i}^{\frac{n+2s}{n-2s}} U_{P^h, \Lambda_h} \\
&= \frac{1}{\lambda} \int_{\mathbb{R}^n} \frac{\partial K}{\partial t_j} \left(\frac{x+P^i}{\lambda}\right) U_{0, \Lambda_i}^{\frac{n+2s}{n-2s}} U_{P^h-P^i, \Lambda_h} - \int_{\mathbb{R}^n} K\left(\frac{x+P^i}{\lambda}\right) U_{0, \Lambda_i}^{\frac{n+2s}{n-2s}} \frac{\partial U_{P^h-P^i, \Lambda_h}}{\partial P_j^i} \\
&= O(\lambda^{-1} \frac{1}{|X^i - X^h|^{n-2s}}) + O(\frac{1}{|X^i - X^h|^{n-s}}).
\end{aligned} \tag{B.23}$$



The first part of (B.2) can be estimated as

$$\begin{aligned}
 \int_{\mathbb{R}^n} \sum_{h \neq i} U_{P^h, \Lambda_h}^{\frac{n+2s}{n-2s}} \left| \frac{\partial U_{P^i, \Lambda_i}}{\partial P_j^i} \right| &\leq \sum_{h \neq i} \int_{\mathbb{R}^n} \frac{C}{(1 + |x - X^h|)^{n+2s}} \frac{1}{(1 + |x - X^i|)^{n-2s+1}} \\
 &\leq \sum_{h \neq i} \frac{C}{|X^h - X^i|^{n-s}} \int_{\mathbb{R}^n} \frac{1}{(1 + |y - X^h|)^{n+s+1}} \\
 &\leq C(\lambda l)^{-(n-s)}.
 \end{aligned} \tag{B.24}$$

Then the expansion of  $\frac{\partial I}{\partial P_j^i}(W_m)$  follows from (B.19), (B.20), (B.21), (B.22), (B.23) and (B.24).  $\square$

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